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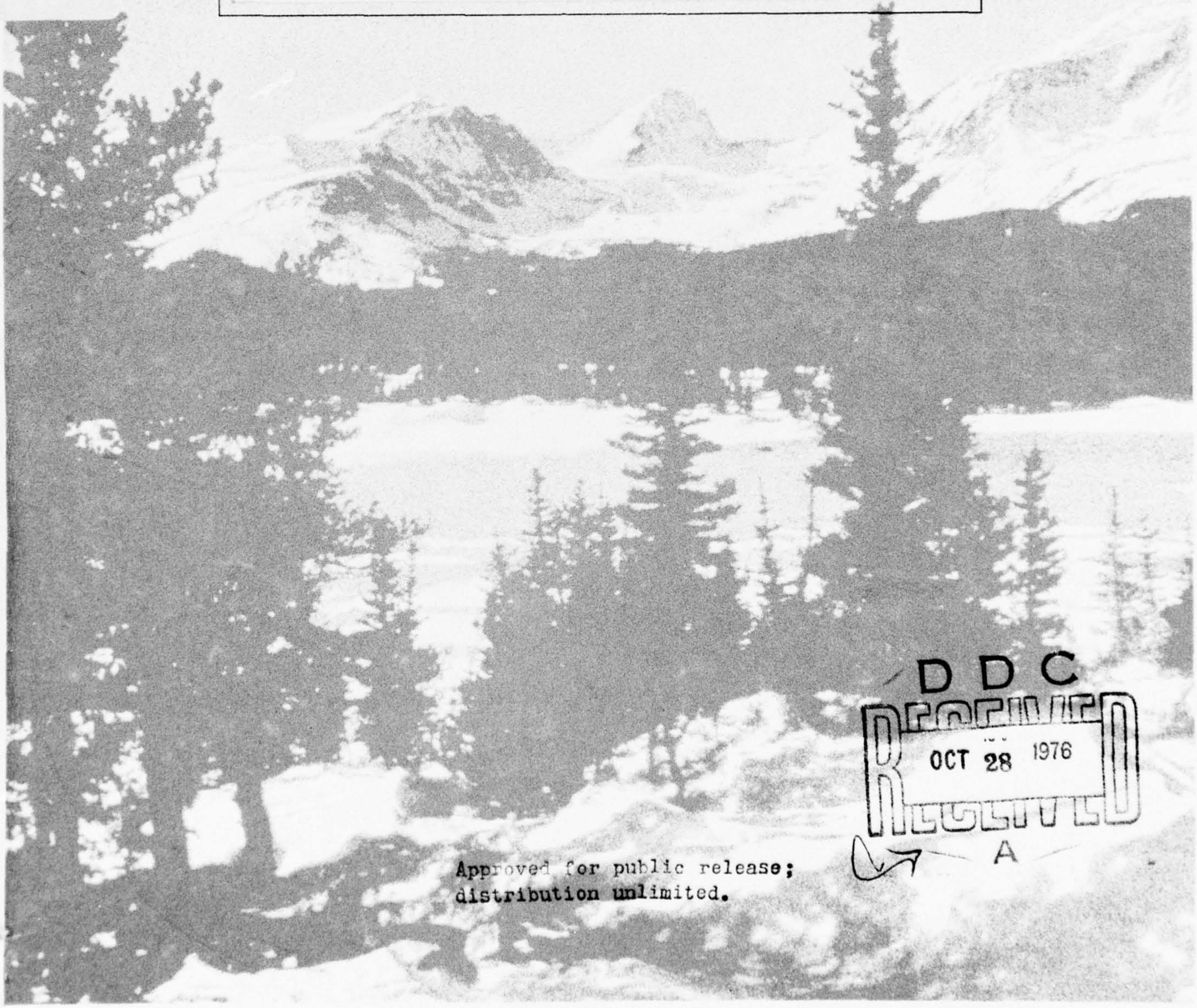

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TRANSPORT PROCESSES WITHIN A PLASMA DRIVEN BY A HIGH-FREQUENCY ELECTRIC FIELD

Dean E. McKinnis

June 1976 (Ph.D. Thesis) CU #1019-A4

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A method is described for calculating transport coefficients within a plasma driven by a high-frequency alternating electric field. The method involves a collision term expressed in the Klimontovich form, and does not require the field to be weak. Representative calculations are made for a nonrelativistic, nondegenerate, two-component, magnetic-field-free plasma by using a large-wave-number functional form to simplify some sums over cont

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ABSTRACT

A method is described for calculating transport coefficients within a plasma driven by a high-frequency alternating electric field. The method involves a collision term expressed in the Klimontovich form, and does not require the field to be weak. Representative calculations are made for a nonrelativistic, nondegenerate, two-component, magnetic-field-free plasma by using a large-wave-number functional form to simplify some sums over wave number; expressions are derived for thermal conductivity, electron resistive heating ("inverse Bremsstrahlung") rate, and interspecies heat exchange rate as functions of electric field strength. Numerical values are given for the three functions.

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CHAPTER I

INTRODUCTION

1. Introduction

This dissertation presents three examples of the use of a collision term derived through the Klimontovich formalism for the calculation of transport coefficients of a plasma driven by a high-frequency electric field which is not necessarily weak. The resistive heating, which has been calculated before,[1] is derived here using the formalism of Goldman (1970), as is the electron-ion thermal energy exchange rate. The collision term as given by Klimontovich and Puchkin (1974) is used as the basis of an expression for the thermal conductivity. The presentation is simplified, in most places, by supposing the ions to be singly charged. In these examples, the effect of an ambient magnetic field is not considered.

Chapter I introduces some topics which will be used in the computations. Chapter II outlines the derivation of the collision term used in the computations. The computation of an inverse Bremsstrahlung rate and a heat-exchange rate from the frequency-domain representation of Goldman is done in Chapter III. The electron thermal conductivity is computed with the use of the Chapman-Enskog

[1]More complete references for these topics are given in the specialized chapters.

method in Chapter IV. The results of the computations of Chapters III and IV are Eqs. (III.41), (III.64), and (IV.71). Their values are shown in Figures 3, 4, and 5 and in Tables I and II; the most important result is probably the expression for the thermal conductivity.

Gaussian c.g.s. units are used. Use of the symbol $y!$ does not imply y an integer, and $\frac{1}{y!}$ is defined by continuity if $y!$ doesn't exist. Many symbols in the text have meanings which are conventional or inferrable from the context, and are not defined when introduced. Most of the symbols used are defined in Chapter VI.

2. Transport Processes in a Plasma

A thermodynamically isolated plasma not in equilibrium will tend to move toward equilibrium. If the departure of the system from an equilibrium configuration is small enough, and the equilibrium configuration is not a special point for the perturbing or restoring agents, then the rate of restoration will be proportional to the perturbation. If an infinitesimal perturbation is maintained by interaction of the plasma with outside forces, the plasma will reach a stationary state in which the perturbation is balanced against the plasma response. The constant of proportionality between the departure from equilibrium and either the perturbing force or the rate of restoration is called a "transport coefficient."

The behavior of systems which deviate infinitesimally from equilibrium is discussed in the books by Prigogine (1967), deGroot (1951), and Cox (1955).

Examples of departure from equilibrium are nonthermal distribution functions, temperature differences between species, macroscopic electric fields, gradients of temperature or concentration. The restoration rate is controlled by interparticle interactions, called "collisions" by extension from the classical billiard-ball molecules. Examples of transport coefficients are electrical conductivity, interspecies equilibration rate, thermal conductivity, rate of relaxation of a single species toward thermal, and diffusivity.

As an example of these ideas, suppose that two populations of greatly differing mass--say ions and electrons--have different temperatures. The coupling between species is weak enough that each population remains essentially thermal, and the temperatures will tend toward equality according to

$$\frac{d\theta_e}{dt} = \frac{\theta_i - \theta_e}{\tau_{ei}} . \tag{1}$$

The electron-ion equilibration time τ_{ei} is a transport coefficient for the process. This is an example of a "transport" process which isn't a flux in configuration space.

Another example of a system with transport is a plasma with imposed temperature gradient and electric field. A stationary state is described by

$$\underline{J} = \sigma \underline{E} + \alpha \nabla \theta \tag{2a}$$

$$\underline{S} = -\beta \underline{E} - K \nabla \theta \tag{2b}$$

giving the electrical current $\underline{J}$ and the heat flux $\underline{S}$ (Spitzer and Härm 1953). The transport processes here are the actual fluxes of

heat and charge, and the transport coefficients are σ, α, β, and K.

In a system with a finite departure from equilibrium, the ratio of rate of restoration to strength of perturbation is likely to be positive, though perhaps not constant. If a periodic electric field is applied to a plasma with species of different temperatures, the species will still exchange thermal energy, though the process is no longer one of equilibration. If the field is strong, it can modify the response to an applied (constant) electric field or temperature gradient; the thermal and electrical conductivity are still constant with respect to the perturbations, but are functions of the driving field. If the plasma response considered is that of the plasma to the driving field itself--the rate of heating by inverse Bremsstrahlung--then the response is nonlinear when the driving field becomes strong.

3. The Effect of a Driving Field

When a plasma is so strongly driven by an electromagnetic field that the resultant electron velocity much exceeds the thermal speed, the most important effect may be the generation of plasma instabilities (DuBois and Goldman 1965; Silin 1965) and electrostatic turbulence (Kadomtsev 1964; Tsytovich 1972). But the turbulence needs time to grow, there may be effects which are independent of turbulence, and it is possible for the driven speed to be near the thermal speed without turbulence (Brownell, Dreicer, Ellis, and Ingraham 1974 and 1975).

An alternating electric field will change the average relative speed of the two species and thereby influence transport processes. The effect will become important as the driven speed of the electron approaches its thermal speed. A good discussion of strong-field effects

is given by Kidder (1974).

The intensity of the electromagnetic wave which drives the electrons at their thermal speed depends on the electron temperature and the wave frequency. It is shown in Figure 1. For that figure, it is supposed that the wave is propagating at the vacuum speed of light; physically, it could be important that the field strength increases in a wave approaching a reflection point. The intensity at which the nominal driven speed reaches the speed of light, so that the treatment given here needs modification, is shown in Figure 2.

4. Integration with respect to Wave Number

The expressions derived in Chapters III and IV involve velocity moments of the collision term (II.29) in the next chapter, and those moments involve integrals over $\underline{k}$-space (and sums over the integers n) of expressions weighted by $\exp\left\{-\frac{1}{2}\left(\frac{n\omega_e}{kv_e}\right)^2\right\}$. For large values of k, the integrands are proportional to k^{-3}. We can expand each integrand in a series

$$I = k^{-3}[I_0 + I_1 + \ldots] \tag{3}$$

and take the leading term. In this process, the cutoff has disappeared that was imposed by the exponential at small values of k, so we re-impose it:

$$\int \frac{d^3k}{(2\pi)^3} I \longrightarrow \int_{k_{min}}^{k_{max}} dk\, k^{-1} \frac{1}{2\pi^2} \int_0^1 d\mu\, I_0 = \frac{\ln\Lambda}{2\pi^2} \int_0^1 d\mu\, I_0 \tag{4}$$

The expression on the end contains the Coulomb logarithm

$$\ln\Lambda \equiv \ln\frac{k_{max}}{k_{min}}, \tag{5}$$

which involves the limits on k. The lower limit k_{min} will be an

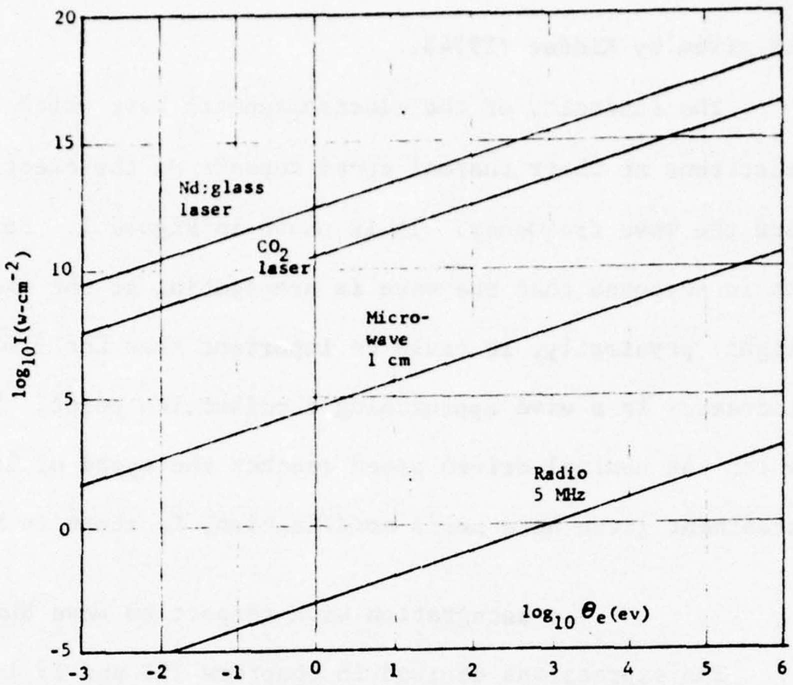

Fig. 1. Intensity at which the speed of an electron driven by a plane-polarized electromagnetic wave reaches thermal speed.

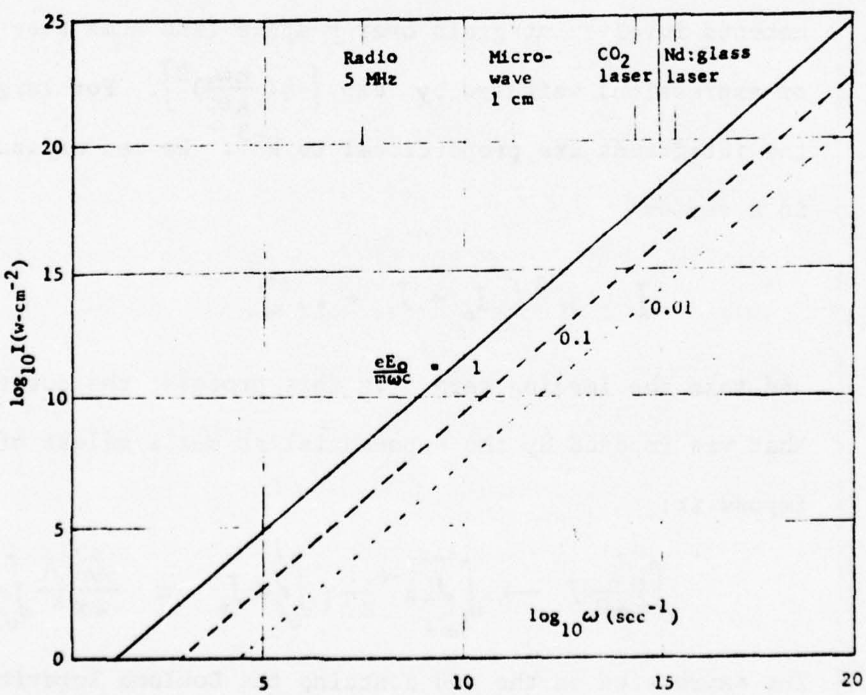

Fig. 2. Relativistic intensity thresholds: Energy flux at which the nominal driven speed of an electron reaches the specified fraction of the speed of light.

average of $\frac{n\omega_0}{\nu_e}$ over the non-negative integers n. Because it enters as the argument of a logarithm, and the logarithm has a value which is typically at least 5 or 10 (Spitzer 1962, p. 128), the logarithm can be estimated by ignoring small multipliers in k_{min}, and we can use

$$k_{min} \approx k_0 \frac{\omega_0}{\omega_p} . \tag{6}$$

The upper limit k_{max} is the inverse of the distance of closest approach b_0, and is required by some terms ignored in the derivation of the collision term in Chapter II. (A brief discussion is given in Appendix F.)

Another form for (4) is

$$\int \frac{d^3k}{(2\pi)^3} k^{-3} \cdots = \frac{m v_e \theta_e}{n e^4 Z} \frac{3}{2(2\pi)^{5/2}} \nu_0 \int_0^1 d\mu \cdots \tag{7}$$

ν_0 is the conventional electron-ion collision frequency for momentum transfer (Perel and Eliashberg 1961; DuBois, Gilinsky, and Kivelson 1963; Kidder 1971):

$$\nu_0 = \frac{4\sqrt{2\pi}}{3} \frac{n_i e^2 e_i^2 \ln\Lambda}{m^{1/2} \theta_e^{3/2}} \tag{8}$$

What happens to Λ in a strong alternating field? The minimum impact parameter decreases as v^{-2}, and the effective quantum-mechanical size of the electron only as v^{-1}, so that at some energy the effective distance of closest approach is determined by the electron size instead of b_0. A parallel argument is given by Marshak (1941). Moreover, the effective speed is some average of the thermal speed and the driven speed; so k_{max} can be calculated by

$$\frac{1}{k_{max}} \approx max(b_o, \lambda\!\!\!^{-}_{deB}), \quad v_{eff} \approx max(v_e, \frac{eE_o}{m\omega_o})$$

$$\lambda_{deB} = \frac{\hbar}{m v_{eff}}, \quad b_o = \frac{e^2}{\frac{1}{2} m v_{eff}^2} \tag{9}$$

The crossover to quantum-mechanical limitation happens when $2\alpha c/v_{eff}$, in which α is the fine-structure constant, decreases past 1; and that happens when the electron energy increases past about 40 ev.

The effect of the strong field on k_{min} can be ignored within our crude approximation, because although the effective shielding distance may be increased by decorrelation of the two species, each species is still shielding itself.

In a plasma with turbulence, there is often important behavior in the spectrum of (e.g., electron density) fluctuations corresponding to propagating Langmuir waves (see, e.g., DuBois and Goldman 1972), and that spectrum is strongly damped unless $k \lesssim 0.3 k_D$. Thus weak turbulence contributes collisional effects described by $\int_0^{k_0} dk$, and non-coleective collisions contribute $\int_{k_0}^{\infty} dk$, where an upper limit may have to be imposed. A variation of this idea is discussed by Kihara and Aono (1963).

5. The Quantities $\Gamma_r(\zeta)$

This section defines and gives some properties of a family of functions that enter the analysis, and which have been given the label Γ_r.

<u>Definition</u>:

$$\Gamma_r(\zeta) = \lim_{\eta \to 0} \sum_{n=-\infty}^{\infty} (\eta n^2)^r e^{-\eta n^2} J_n^2(\zeta\sqrt{\tfrac{2}{\eta}}) \quad r = 0,1,2 \ldots \tag{10}$$

Expression for Γ_r:

A derivation of an expression for Γ_r is given here. (Another derivation is given in Appendix A.) The starting point is the definition of Γ_o:

$$\Gamma_o(\zeta) = \lim_{\eta\to 0} \sum_n e^{-\eta n^2} J_n^2(\zeta\sqrt{\tfrac{2}{\eta}}). \tag{11}$$

An easily derived identity (Watson 1944, p. 32) is

$$J_n^2(\zeta\sqrt{\tfrac{2}{\eta}}) = \int_0^{\pi/2} \frac{d\theta}{\pi/2}\, J_o(2\zeta\sqrt{\tfrac{2}{\eta}}\sin\theta)\cos 2n\theta \tag{12}$$

For (fixed) nonzero ζ, the contribution from the J_o term will come only from arguments

$$\theta\eta^{-\frac{1}{2}} \leq \mathcal{O}(1). \tag{13}$$

When η is small enough, the contributing part is a smooth function of n, and the sum is tractable:

$$x \equiv n\eta^{1/2},\ \sum_n \longrightarrow \int\frac{dx}{\sqrt{\eta}} \tag{14}$$

$$\Gamma_o(\zeta) = \lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{dx}{\sqrt{\eta}}e^{-x^2}\int_0^{\infty}\frac{d\theta}{\pi/2}\cos(\frac{2\theta x}{\sqrt{\eta}})\int_0^{\infty}\frac{d\alpha}{2\pi}e^{-i\zeta 2\sqrt{\frac{2}{\eta}}\theta\sin\alpha} \tag{15}$$

in which an integral representation for J_o is used.

$$\int_{-\infty}^{\infty}dx\, e^{-x^2}\cos(2\theta x\eta^{-\frac{1}{2}}) = \pi^{\frac{1}{2}}e^{-\theta^2/\eta} \tag{16}$$

$$\Gamma_o(\zeta) = \int_0^{2\pi}\frac{d\alpha}{2\pi}\lim_{\eta\to 0}\eta^{-\frac{1}{2}}\int_0^{\infty}\frac{d\theta}{\pi/2}\pi^{1/2}e^{-\theta^2/\eta}e^{-i\zeta 2\sqrt{\frac{2}{\eta}}\theta\sin\alpha} \tag{17}$$

Γ_o is real, so $2\int_0^{\infty}d\theta \rightarrow \int_{-\infty}^{\infty}d\theta$.

$$\Gamma_o(\zeta) = \int_0^{2\pi}\frac{d\alpha}{2\pi}\lim_{\eta\to 0}\eta^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{d\theta}{\pi}\pi^{1/2}\exp[-(\theta\eta^{-\frac{1}{2}}+i\sqrt{2}\zeta\sin\alpha)^2-2\zeta^2\sin^2\alpha] \tag{18}$$

$$= e^{-\zeta^2}\int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{\zeta^2\cos 2\alpha} \tag{19}$$

$$= \Lambda_0(\zeta^2) \tag{20}$$

in which a modified Bessel function appears:

$$\Lambda_n(x) = e^{-x} I_n(x) \tag{21}$$

An expression for the other Γ's can be obtained by including a factor β in the argument of the exponential in (11). The value of the expression becomes $\Lambda_0(\beta\zeta^2)$, so that

$$\Gamma_n(\zeta) = \left[(-)^n\left(\frac{\partial}{\partial\beta}\right)^n \Gamma_0(\zeta\sqrt{\beta})\right]_{\beta=1} \tag{22}$$

$$= (-\zeta^2)^n \Lambda_0^{(n)}(\zeta^2) \tag{23}$$

From Abramowitz and Stegun (1964, p. 377), we have

$$\Lambda_0(\zeta^2) = M(\tfrac{1}{2}, 1, -2\zeta^2) \tag{24}$$

and also that

$$\Lambda_0^{(n)}(\zeta^2) = (-2)^n \frac{(n-\frac{1}{2})!}{n!\,(-\frac{1}{2})!} M(n+\tfrac{1}{2}, n+1, -2\zeta^2) \tag{25}$$

to give the result

$$\Gamma_n(\zeta) = \frac{(n-\frac{1}{2})!}{n!\,(-\frac{1}{2})!}(2\zeta^2)^n M(n+\tfrac{1}{2}, n+1, -2\zeta^2) \tag{26}$$

The symbol M is the confluent hypergeometric function (Jahnke and Emde 1945, p. 275).

A related quantity:

The expression preceding the equality sign in (33) below enters the analysis. It can be signified by the change of variables

$$\eta = \frac{1}{2}\left(\frac{\omega_o}{k v_e}\right)^2 \tag{27}$$

$$\zeta = \mu E \tag{28}$$

in which a nondimensionalized field strength is introduced:

$$E = \frac{v_E}{2v_e} = \frac{eE_o}{2m\omega_o v_e}. \tag{29}$$

The argument of the Bessel functions is

$$\underline{k}\cdot\underline{d}_e = k\mu\frac{v_E}{\omega_o} = 2Ev_e\frac{k\mu}{\omega_o} = \zeta\sqrt{\frac{2}{\eta}}. \tag{30}$$

The susceptibility term is

$$\mathrm{Im}\,\chi_e(n\omega_o) = \frac{k_{De}^2}{k^2}\pi^{1/2}\frac{n\omega_o}{kv_e\sqrt{2}}e^{-\left(\frac{n\omega_o}{kv_e\sqrt{2}}\right)^2} \tag{31}$$

$$= \frac{k_{De}^2}{k^2}\pi^{1/2}n\sqrt{\eta}\,e^{-\eta n^2} \tag{32}$$

Now the expression of interest:

$$\sum_n\left(\frac{n\omega_o}{kv_e\sqrt{2}}\right)^{2r-1}\mathrm{Im}\,\chi_e(n\omega_o)\,J_n^2(\underline{k}\cdot\underline{d}_e)$$

$$= \sum_n(\eta n^2)^r\frac{k_{De}^2}{k^2}\pi^{1/2}e^{-\eta n^2}J_n^2\left(\zeta\sqrt{\frac{2}{\eta}}\right) \tag{33}$$

Operating on that expression with $k^{-2}\lim_{k\to\infty}k^2$ gives the result

$$k^{-2}\lim_{k\to\infty}k^2\sum_{n=-\infty}^{\infty}\left(\frac{n\omega_o}{kv_e\sqrt{2}}\right)^{2r-1}\mathrm{Im}\,\chi_{e\underline{k}}(n\omega_o)\,J_n^2(\underline{k}\cdot\underline{d}_e)$$

$$= \left(\frac{k_D}{k}\right)^2\pi^{1/2}\Gamma_r(\zeta). \tag{34}$$

Some properties of Γ_r:

$\Gamma_r(\zeta)$ can be expressed in terms of modified Bessel functions by a symmetric finite sum (Appendix B):

$$\Gamma_r(\zeta) = \frac{(r-\frac{1}{2})!}{r!\,(-\frac{1}{2})!}(2\zeta^2)^r \sum_{n=-\infty}^{\infty} (-)^n \Lambda_n(\zeta^2)\,\frac{r!\,r!}{(r+n)!\,(r-n)!} \tag{35}$$

A recursion relation is derived in Appendix C:

$$2\zeta^2\,\Gamma_r(\zeta) = (2r-1)\,\zeta^2\,\Gamma_{r-1} - r\,\Gamma_r + \Gamma_{r+1} \tag{36}$$

Appendix C also gives a differential recursion formula:

$$\zeta\,\Gamma_r'(\zeta) = 2r\,\Gamma_r - 2\,\Gamma_{r+1} \tag{37}$$

Small argument:

$$\Gamma_r(\zeta) = \frac{(2r-1)!!}{r!}\,\zeta^{2r}\left[1 - \frac{2r+1}{r+1}\zeta^2 + \frac{1}{2!}\,\frac{2r+1}{r+1}\,\frac{2r+3}{r+2}\,\zeta^4 - \ldots\right] \tag{38}$$

For large argument, Jahnke and Emde (1945) give

$$M(\alpha, \gamma, -2\zeta^2) \sim \frac{(\gamma-1)!}{(\gamma-\alpha-1)!}(2\zeta^2)^{-\alpha}\left[1 + \alpha\,\frac{\alpha-\gamma+1}{2\zeta^2} + \ldots\right] \tag{39}$$

so that

$$\Gamma_r(\zeta) \sim \frac{(r-\frac{1}{2})!}{(-\frac{1}{2})!}\sqrt{\frac{2}{\pi}}\;\zeta^{-1}\left[1 + \frac{2r+1}{8\zeta^2} + \ldots\right]. \tag{40}$$

CHAPTER II

THE COLLISION TERM

The evolution in time of a particle distribution function obeys the Boltzmann equation

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{\partial}{\partial \underline{v}} \cdot \vec{a} f = \left(\frac{\partial f}{\partial t}\right)_{coll}. \tag{1}$$

The right-hand side, the "collision term," is that part of $\frac{\partial f}{\partial t}$ which is generated by interparticle interactions. In this chapter, expressions are developed following the treatment of Goldman (1970) for two velocity moments of the collision term

$$I_{\alpha \underline{k}}(\omega,\underline{v}) \equiv \left(\frac{\partial f_\alpha}{\partial t}\right)_{coll}(\underline{k},\omega,\underline{v}) \tag{2}$$

in which the Fourier-transform variables $\underline{k}$ and ω appear. An expression given by Klimontovich and Punchkin (1974) for the time-dependent form is noted for later use.

1. Development in the Time Domain

A specified set of identical point particles can be described by the exact ("Klimontovich") distribution function

$$f^{Mic}(\vec{x},\vec{v},t) = \sum_{i=1}^{N} \delta[\vec{x}-\vec{x}_i(t)]\,\delta[\vec{v}-\vec{v}_i(t)] \tag{3}$$

where $(\vec{x}_i(t), m\vec{v}_i(t))$ is the position of the i^{th} of the N particles in a (six-dimensional) single-particle phase space. It has the

exact equation of motion

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\frac{\partial}{\partial \underline{x}}\right) f^{Mic}(\underline{x},\underline{v},t) + \frac{\partial}{\partial \underline{v}}\cdot\left[m\,\underline{a}^{Mic}(\underline{x},\underline{v},t)\, f^{Mic}\right] = \mathcal{O}(\hbar). \quad (4)$$

This equation is useful if the right-hand side is set to zero. $\underline{a}^{\mathrm{Mic}}(\underline{x},\underline{v},t)$ is the acceleration that the field at $(\underline{x},\underline{v},t)$ would give a particle there; the self-field of the particle is to be neglected. A parallel development using the wave equation and a semiclassical treatment of the field seems feasible (Rand 1964; Brown and Kibble 1964; Pert 1972; Seely and Harris 1973; Seely 1974), but is more cumbersome in describing classical particles.

The system can be considered to be a member of a Gibbs microcanonical ensemble, and averages taken over the ensemble. We will use the notation suggested by

$$f(\underline{x},\underline{v},t) = \langle f^{Mic}(\underline{x},\underline{v},t)\rangle, \quad (5a)$$

$$\delta f(\underline{x},\underline{v},t) = f^{Mic} - f. \quad (5b)$$

Taking the ensemble average of each term of (4) with $\hbar = 0$ leads to

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\frac{\partial}{\partial \underline{x}}\right) f + \frac{\partial}{\partial \underline{v}}\cdot\underline{a}f = -\frac{\partial}{\partial \underline{v}}\cdot\langle \delta\underline{a}(\underline{x}\underline{v}t)\,\delta f(\underline{x}\underline{v}t)\rangle, \quad (6)$$

and a problem in kinetic theory is the evaluation of the right-hand side. The Boltzmann collision integral and that of Landau are based on the picture of a binary collision, and a statistical argument leads to the Fokker-Planck form for (6) (Chapman and Cowling 1939, p. 46 ff; Chandrasekhar 1943; Wu 1966, p. 32; Present 1958; Landau 1936; Krall and Trivelpiece 1973; Delcroix 1960, p. 116).

The difference of (4) and (6) is

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\frac{\partial}{\partial \underline{x}}\right)\delta f(\underline{x}\underline{v}t) + \frac{\partial}{\partial \underline{v}}\cdot\underline{a}\,\delta f = -\frac{\partial}{\partial \underline{v}}\cdot\left[f\delta\underline{a} + \delta f\delta\underline{a} - \langle\delta f\delta\underline{a}\rangle\right]. \quad (7)$$

We specialize now to a gas of particles coupled only by electromagnetic forces; then $\underline{a}$ commutes with $\frac{\partial}{\partial \underline{v}}$. We ignore terms quadratic in the fluctuations. Allowing more than one kind of particle in the system and introducing a species subscript α gives the equations

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\frac{\partial}{\partial \underline{x}} + \underline{a}_\alpha\cdot\frac{\partial}{\partial \underline{v}}\right)\delta f_\alpha(\underline{x}\underline{v}t) = S_\alpha(\underline{x}\underline{v}t) \equiv -\delta\underline{a}_\alpha\cdot\frac{\partial f_\alpha}{\partial \underline{v}}, \quad (8)$$

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\frac{\partial}{\partial \underline{x}} + \underline{a}_\alpha\cdot\frac{\partial}{\partial \underline{v}}\right)f_\alpha(\underline{x}\underline{v}t) = I_\alpha(\underline{x}\underline{v}t) \equiv -\langle\delta\underline{a}_\alpha\cdot\frac{\partial \delta f_\alpha}{\partial \underline{v}}\rangle. \quad (9)$$

The treatment in this chapter follows that of Goldman (1970).

A Green's function for the operator in (8) is

$$\Theta(t-t')G_\alpha(\underline{x}\underline{x}'\underline{v}\underline{v}'tt') = \Theta(t-t')\delta[\underline{x}-\underline{x}^\circ_\alpha(t;\underline{x}'\underline{v}'t')]\,\delta[\underline{v}-\underline{v}^\circ_\alpha(t;\underline{x}'\underline{v}'t')] \quad (10)$$

in which $\underline{x}^o_\alpha$ and $\underline{v}^o_\alpha$ are the coordinates a particle would have at time t if it began at ($\underline{x}'\underline{v}'t'$) and moved with acceleration $\underline{a}_\alpha(\underline{x},\underline{v},t)$:

$$\underline{v}^\circ_\alpha(t;\underline{x}'\underline{v}'t') = \underline{v}' + \int_{t'}^{t} d\tau\, \underline{a}_\alpha(\underline{x}(\tau),\underline{v}(\tau),\tau) \quad (11)$$

$$\underline{x}^\circ_\alpha(t;\underline{x}'\underline{v}'t') = \underline{x}' + \underline{v}'_\alpha(t-t') + \int_{t'}^{t} d\tau_1 \int_{t'}^{\tau_1} d\tau_2\, \underline{a}_\alpha(\underline{x}(\tau_2),\underline{v}(\tau_2),\tau_2) \quad (12)$$

G_α has the property

$$G_\alpha(\underline{x}\underline{x}'\underline{v}\underline{v}'tt') = \delta(\underline{x}-\underline{x}')\,\delta(\underline{v}-\underline{v}'). \quad (13)$$

The formal solution of (8) which takes a specified value at $t = t_o$ is

$$\delta f_\alpha(\vec{x}\vec{v}t) = \int d^3x'\, d^3v'\, \delta f_\alpha(\vec{x}'\vec{v}'t_0)\, G_\alpha(\vec{x}\vec{x}'\vec{v}\vec{v}'tt_0)$$

$$+ \int d^3x'\, d^3v' \int_{t_0}^{\infty} dt'\, \Theta(t-t')\, S_\alpha(\vec{x}'\vec{v}'t')\, G_\alpha(\vec{x}\vec{x}'\vec{v}\vec{v}'tt') \qquad (14)$$

This is not an explicit solution because the source term S_α depends on δf_α through $\delta \underline{a}_\alpha$. The dependence is expressed by Poisson's equation

$$\nabla \cdot \delta \underline{E}(\underline{x}t) = 4\pi\, \delta\rho(\underline{x}t) \equiv 4\pi \sum_{\alpha=e,i} \delta\rho_\alpha(\underline{x},t)$$

$$= 4\pi \sum_\alpha q_\alpha \int d^3v\ \ \delta f_\alpha(\underline{x}\underline{v}t). \qquad (15)$$

If we define a homogeneous plasma as one for which the ensemble averages $f_\alpha(\underline{x}\underline{p}t)$ and $\underline{a}_\alpha(\underline{x}\underline{p}t)$ are independent of $\underline{x}$, and suppose our system to be such a plasma, then G depends on $\underline{x}$ and $\underline{x}'$ only through the difference $\underline{x}-\underline{x}'$, and we can take space Fourier transforms:

$$\delta f_{\alpha\underline{k}}(\vec{v}t) \equiv \int d^3x\, e^{-i\underline{k}\cdot\underline{x}}\, \delta f(\underline{x}\underline{v}\, t) \qquad (16)$$

$$G_{\alpha k}(\vec{v}\vec{v}'tt') \equiv \int d^3(\underline{x}-\underline{x}')\, e^{-i\underline{k}\cdot(\underline{x}-\underline{x}')}\, G_\alpha(\underline{x}\underline{x}'\underline{v}\underline{v}'tt') \qquad (17)$$

The quantity $\delta\underline{E}_{\underline{k}}(t)$ has an important property: $\delta\underline{E}(\underline{x},t)$ is the field of a set of point particles moving at nonrelativistic speeds, so that interaction is effectively instantaneous:

$$\delta\underline{E}_{\underline{k}}(t) = \int d^3x\, e^{-i\underline{k}\cdot\underline{x}} \sum_i q_i \frac{\vec{x}-\vec{x}_i}{|\vec{x}-\vec{x}_i|^3}$$

$$= \sum_i q_i\, e^{-i\underline{k}\cdot\underline{x}_i} \int d^3x\, \frac{\vec{x}}{x^3}\, e^{-i\underline{k}\cdot\underline{x}}, \qquad (18)$$

and that is parallel to $\underline{k}$, so that

$$\delta \underline{E}_{\underline{k}}(t) = \hat{k}\,\delta E_{\underline{k}}(t). \tag{19}$$

This defines the signed quantity $\delta E_{\underline{k}}(t)$.

We use the abbreviations

$$\Delta_\alpha tt' = \int_{t'}^{t} d\tau\, \underline{a}_\alpha(\tau) \tag{20}$$

$$\varphi_{\alpha\underline{k}}(\underline{v}'tt') = \vec{k}\cdot\left[\underline{v}'(t-t') + \int_{t'}^{t} d\tau\,\Delta_\alpha \tau t'\right] \tag{21}$$

Some Fourier transforms are listed:

$$G_{\alpha\underline{k}}(\underline{v}\underline{v}'tt') = \delta(\underline{v}-\underline{v}'-\Delta tt')e^{-i\varphi_{\alpha\underline{k}}(\underline{v}'tt')} \tag{22}$$

$$= \delta(\underline{v}-\underline{v}'-\Delta tt')e^{-i\varphi_{\alpha k}(\underline{v}-\Delta tt', t, t')} \tag{23}$$

$$\delta f^{\circ}_{\alpha\underline{k}}(\underline{v}t) \equiv \int d^3v'\,\delta f_{\alpha\underline{k}}(\underline{v}'t_\circ)\,G_{\alpha\underline{k}}(\underline{v}\underline{v}'tt_\circ) \tag{24}$$

$$G_{\alpha\underline{k}}(\underline{v}\underline{v}'tt') = \delta(\vec{v}-\vec{v}') \tag{25}$$

$$\delta f_{\alpha\underline{k}}(\underline{v}t) = \delta f^{\circ}_{\alpha\underline{k}}(\underline{v}t) + \int d^3v'\int_{t_\circ}^{\infty} dt'\,\Theta(t-t')\,G_{\alpha\underline{k}}(\underline{v}\underline{v}'tt')\,S_{\alpha\underline{k}}(\underline{v}'t') \tag{26}$$

$$S_{\alpha\underline{k}}(\underline{v}t) = -\delta\underline{a}_{\alpha k}(t)\cdot\frac{\partial}{\partial \underline{v}} f_\alpha(\vec{v}t) = -q_\alpha\,\delta E_{\underline{k}}(t)\,\hat{k}\cdot\frac{\partial f_\alpha(\vec{v}t)}{\partial \underline{v}} \tag{27}$$

$$\delta E_k(t) = \frac{4\pi}{ik}\,\delta\rho_{\underline{k}}(t) \qquad (k \neq 0) \tag{28}$$

It can be seen from (18) that $\delta E_{\underline{k}}$ vanishes at $\underline{k}=0$.

In terms of $\underline{k}$-dependent variables, the collision term from (9) is

$$I_\alpha(\vec{v},t) = -\frac{\partial}{\partial\vec{v}}\cdot\sum_{\underline{k}} q_\alpha \hat{k}\,\langle \delta E_{\underline{k}}(t)\,\delta f_{\alpha,-\underline{k}}(\vec{v}.t)\rangle. \tag{29}$$

2. Symbolic Solution for Correlation Functions

In this section, expressions for $\langle \delta E_k(t) \delta E_{-k}(t') \rangle$ and for $\langle \delta E_{\underline{k}}(t) \, \delta f_{\alpha, -\underline{k}}(\vec{v} t) \rangle$ are written down.

Following are some definitions for some symbols used.

The nonlinear susceptibility $\chi_{\alpha\underline{k}}(t,t')$:

$$-\frac{ik}{4\pi}\chi_{\alpha\underline{k}}(tt') = -\frac{q_\alpha^2}{m_\alpha}\int d^3v\, d^3v'\, \Theta(t-t')\, G_{\alpha\underline{k}}(\vec{v}\vec{v}'tt')\hat{k}\cdot \frac{\partial f_\alpha(v't')}{\partial \vec{v}'} \tag{30}$$

The charge density fluctuation for one species:

$$\delta\rho_{\alpha\underline{k}}(t) \equiv \int d^3v\, q_\alpha\, \delta f_{\alpha\underline{k}}(\vec{v}t) \tag{31}$$

The part of the charge density fluctuation which has evolved from the initial value:

$$\delta\rho^{\circ}_{\alpha\underline{k}}(t) \equiv \int d^3v\, q_\alpha\, \delta f^{\circ}_{\alpha k}(\vec{v}\, t) \tag{32}$$

Total susceptibility and charge density:

$$\chi_{\underline{k}} \equiv \chi_{e\underline{k}} + \chi_{i\underline{k}}, \qquad \delta\rho_{\underline{k}} = \delta\rho_{e\underline{k}} + \delta\rho_{i\underline{k}} \tag{33}$$

The dielectric function:

$$\epsilon_{\underline{k}}(tt') \equiv \delta(t-t') + \chi_{\underline{k}}(tt') \tag{34}$$

The inverse dielectric function:

$$\int dt''\, \epsilon^{-1}_{\underline{k}}(tt'')\, \epsilon_{\underline{k}}(t''t') = \delta(t-t') \tag{35}$$

The last equation defines $\epsilon^{-1}_{\underline{k}}(tt')$; the questions of existence and uniqueness are deferred.

Development of the expressions for the defined quantities:

Using (26) in (31) gives

$$\delta\rho_{\alpha\underline{k}}(t) = \delta\rho^{\circ}_{\alpha k}(t) - \frac{ik}{4\pi}\int_{t_o}^{\infty} dt'\, \chi_{\alpha\underline{k}}(tt')\, \delta E_{\underline{k}}(t') \tag{36}$$

$$ik\chi_{\alpha\underline{k}}(tt') = \frac{\omega^2_{p\alpha}}{n_\alpha}\Theta(t-t')\int d^3v\, d^3v'\, G_{\alpha\underline{k}}(\vec{v}\vec{v}'tt')\hat{k}\cdot\frac{\partial f_\alpha(\underline{v}'t')}{\partial \underline{v}'} \tag{37}$$

$$= \frac{\omega^2_{p\alpha}}{n_\alpha}\Theta(t-t')\int d^3v\, \hat{k}\cdot\frac{\partial f_\alpha(\vec{v}-\Delta tt',t')}{\partial\vec{v}} e^{-i\varphi_{\alpha\underline{k}}(\vec{v}-\Delta_\alpha tt',t,t')} \tag{38}$$

If t_o is chosen to be recent enough that collisions haven't had time to change the distribution function very much--i.e., if the right-hand side of (6) is set to zero--then f_α is explicitly known from its initial value:

$$f_\alpha(\vec{v}t) = f_\alpha(\vec{v}-\Delta_\alpha tt_o, t_o) \tag{39}$$

If the initial distribution was Maxwellian, this condition may be applicable even if collisions have had time to be effective, because the fastest effect of collisions is intraspecies thermalization. Then using $f_\alpha(\underline{v}-\Delta tt', t') = f_\alpha(v-\Delta tt_o, t_o)$ in (38) gives

$$ik\chi_{\alpha\underline{k}}(tt') = \frac{\omega^2_{p\alpha}}{n_\alpha}\Theta(t-t')\int d^3v\, \hat{k}\cdot\frac{\partial f_\alpha(\vec{v},t_o)}{\partial\vec{v}} e^{-i\varphi_{\alpha\underline{k}}(v+\Delta t' t_o, t, t')} \tag{40}$$

The expression for E, from (28), is

$$\delta E_{\underline{k}}(t) = \frac{4\pi}{ik}\left[\delta\rho^{\circ}_{\underline{k}}(t) - \frac{ik}{4\pi}\int_{t_o}^{\infty} dt'\,\chi_k(tt')\,\delta E_k(t')\right] \tag{41}$$

which is the same as

$$\int dt'\,\epsilon_{\underline{k}}(tt')\,\delta E_{\underline{k}}(t') = \frac{4\pi}{ik}\,\delta\rho^{\circ}_{\underline{k}}(t), \tag{42}$$

so that an explicit expression for δE is

$$\delta E_{\underline{k}}(t) = \int dt' \, \epsilon^{-1}_{\underline{k}}(tt') \frac{4\pi}{ik} \delta\rho^{\circ}_{\underline{k}}(t'). \tag{43}$$

We assume that $f_{\alpha}(\underline{v}t_o)$ is known. Then we can use the expression derived in Appendix D following Wu (1967):

$$\langle \delta f_{\alpha\underline{k}}(\vec{v}t_o)\, \delta f_{\beta,-\underline{k}}(\vec{v}_1 t_o)\rangle = \delta_{\alpha\beta}\, \delta(\vec{v}-\vec{v}_1)\, f_{\alpha}(\vec{v}t_o) \tag{44}$$

This quantity is independent of $\underline{k}$. In Appendix E, we prove the identity

$$\int d^3v''\, G_{\alpha\underline{k}}(\underline{v}\underline{v}''tt_o)\, G_{\alpha,-k}(\underline{v}'\underline{v}''t't_o) = G_{\alpha\underline{k}}(\underline{v}\underline{v}'tt'). \tag{45}$$

For the field correlation function, we develop $\langle \delta\rho^{\circ}\delta\rho^{\circ}\rangle$:

$$\begin{aligned}
&\langle \delta f^{o}_{\alpha\underline{k}}(\underline{v}_1 t_1)\, \delta f^{o}_{\beta,-\underline{k}}(\underline{v}t)\rangle \\
&= \int d^3v_1'\, d^3v' \langle \delta f_{\alpha\underline{k}}(\underline{v}_1' t_o)\, \delta f_{\beta,-\underline{k}}(\underline{v}' t_o)\rangle\, G_{\alpha\underline{k}}(\underline{v}_1\underline{v}_1' t_1 t_o)\, G_{\beta,-\underline{k}}(\underline{v}\underline{v}'tt_o) \\
&= \int d^3v'\, \delta_{\alpha\beta}\, f_{\alpha}(\vec{v}-\Delta_{\alpha}tt_o, t_o)\, G_{\alpha k}(\vec{v}_1\vec{v}'t_1 t_o)\, G_{\alpha,-k}(\vec{v}\vec{v}'tt_o) \\
&= \delta_{\alpha\beta}\, f_{\alpha}(\underline{v}-\Delta_{\alpha}tt_o, t_o)\, G_{\alpha\underline{k}}(\underline{v}_1\underline{v}t_1 t).
\end{aligned} \tag{46}$$

Then

$$\begin{aligned}
\langle \delta\rho^{\circ}_{\underline{k}}(t_1)\, \delta f^{\circ}_{\alpha,-\underline{k}}(\underline{v}t) &= \int d^3v_1\, q_{\alpha}\, f_{\alpha}(\underline{v}-\Delta_{\alpha}tt_o, t_o)\, G_{\alpha\underline{k}}(\underline{v}_1\underline{v}t_1 t) \\
&= q_{\alpha}\, f_{\alpha}(\underline{v}-\Delta_{\alpha}tt_o, t_o)\, e^{-i\varphi_{\alpha\underline{k}}(\underline{v}t_1 t)}
\end{aligned} \tag{47}$$

$$\langle \delta\rho_{\underline{k}}^{\circ}(t)\,\delta\rho_{-\underline{k}}^{\circ}(t')\rangle = \sum_{\alpha} q_{\alpha}^{2}\int d^{3}v\, f_{\alpha}(vt_{o})\, e^{-i\varphi_{\alpha\underline{k}}(v+\Delta_{\alpha}t't_{o},t,t')} \tag{48}$$

This expression into (40) gives

$$\langle \delta E_{\underline{k}}(t)\,\delta E_{-\underline{k}}(t')\rangle$$
$$= \int dt''dt'''\,\epsilon_{\underline{k}}^{-1}(tt'')\,\epsilon_{-\underline{k}}^{-1}(t't''')\left(\frac{4\pi}{ik}\right)^{2}\int d^{3}v\sum_{\alpha} q_{\alpha}^{2} f_{\alpha}(\underline{v}t_{o})\, e^{-i\varphi_{\alpha k}(v+\Delta_{\alpha}t'''t_{o},t'',t''')} \tag{49}$$

The term $\langle \delta E\delta f\rangle$ can be expressed in terms of this. Making the substitutions suggested by

$$\delta E = \epsilon^{-1}\delta\rho^{\circ},\quad \delta f = \delta f^{\circ} + G\delta E,\quad \delta E\delta f = \epsilon^{-1}\delta\rho^{\circ}\delta f^{\circ} + G\delta E\delta E$$

gives

$$\langle \delta E_{k}(t)\,\delta f_{\alpha,-\underline{k}}(\underline{v}t')\rangle = \int dt'\,\epsilon_{\underline{k}}^{-1}(t,t')\,\frac{4\pi}{ik}\langle \delta\rho_{\underline{k}}^{\circ}(t')\,\delta f_{\alpha,-\underline{k}}^{\circ}(vt)\rangle$$
$$+\int d^{3}v'\int_{t_{o}}^{\infty} dt'\,\Theta(t-t')\,G_{\alpha,-\underline{k}}(\underline{v}\underline{v}'tt')\,q_{\alpha}\hat{k}\cdot\frac{\partial f_{\alpha}(v't')}{\partial \underline{v}'}\langle \delta E_{\underline{k}}(t)\,\delta E_{-\underline{k}}(t')\rangle \tag{50}$$
$$= \int dt'\,\epsilon_{\underline{k}}^{-1}(tt')\,\frac{4\pi}{ik}\, q_{\alpha} f_{\alpha}(\underline{v}-\Delta tt_{o}, t_{o})\, e^{-i\varphi_{\alpha\underline{k}}(v,t't)}$$
$$+\int_{t_{o}}^{\infty} dt'\,\Theta(t-t')\, q_{\alpha}\hat{k}\cdot\frac{\partial f_{\alpha}(\underline{v}-\Delta tt',t')}{\partial \underline{v}}\, e^{-i\varphi_{\alpha,-\underline{k}}(\underline{v}-\Delta tt',t,t')}\langle \delta E_{k}(t)\,\delta E_{-k}(t')\rangle \tag{51}$$

3. Specialization of the Driving Field

For the examples given in Chapters III and IV, we specialize to a system in a linearly polarized electric field with no magnetic field:

$$m_{\alpha}\,\underline{a}_{\alpha}(\underline{v},t) = q_{\alpha}\,\underline{E}_{o}\sin\omega_{o}t. \tag{52}$$

It is turned on after time t_o in such a way that for $t > 0$,

$$\Delta_{\alpha}tt_{o} = \int_{t_{o}}^{t} d\tau\,\underline{a}_{\alpha}(\underline{v},\tau) = -\underline{v}_{o\alpha}\cos\omega_{o}t \tag{53}$$

and

$$\int_{t_0}^{t} dt' \Delta_\alpha t' t_0 = -\underline{d}_\alpha \sin\omega_0 t. \tag{54}$$

In this, $\underline{u}_{0\alpha} = \dfrac{e_\alpha E_0}{m_\alpha \omega_0}$, $\underline{d}_\alpha = \dfrac{\underline{u}_{0\alpha}}{\omega_0}$.

The forms taken by (20) and (21) are

$$\Delta_\alpha t t' = \Delta_\alpha t t_0 - \Delta_\alpha t' t_0 = -\underline{u}_{0\alpha}(\cos\omega_0 t - \cos\omega_0 t') \tag{55}$$

$$\begin{aligned} \int_{t'}^{t} d\tau \Delta_\alpha \tau t' &= \int_{t'}^{t} d\tau (\Delta_\alpha \tau t_0 - \Delta_\alpha t' t_0) \\ &= -\underline{d}_\alpha (\sin\omega_0 t - \sin\omega_0 t') + (t-t')\underline{u}_{0\alpha} \cos\omega_0 t' \end{aligned} \tag{56}$$

$$\varphi_{\alpha \underline{k}}(\underline{v}' t t') = \underline{k} \cdot \left[(\underline{v}' + \underline{u}_{0\alpha} \cos\omega_0 t')(t-t') - \vec{d}_\alpha (\sin\omega_0 t - \sin\omega_0 t') \right] \tag{57}$$

The susceptibility (40) is

$$ik\chi_{\alpha k}(tt') = \frac{\omega_{p\alpha}^2}{n_\alpha} \Theta(t-t') \int d^3v \, \hat{k} \cdot \frac{\partial f(\underline{v} t_0)}{\partial \underline{v}} e^{-i\underline{k}\cdot[\underline{v}'(t-t') - \underline{d}_\alpha(\sin\omega_0 t - \sin\omega_0 t')]} \tag{58}$$

4. Fourier-Transformed Expressions

This section gives a sketch of the derivation of two moments of the collision term for later use. If the system is stationary except for the periodicity imposed by $\underline{Q}(\underline{v},t)$ periodic with period ω_0, then any function with two time arguments f(t,t') determined by the system can be expressed in the form g(t-t',t'), where g is periodic in the second argument. A double Fourier expansion can be made:

$$f(t,t') = \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} e^{-i\ell\omega_0 t'} f(\omega, \ell\omega_0 - \omega) \tag{59}$$

$$f(\omega,\omega') = \int dt\, dt'\, e^{i\omega t} e^{i\omega' t'} f(t,t') \tag{60}$$

$$\varphi(t) = \sum_{\ell} \varphi(\ell\omega_0) e^{-i\ell\omega_0 t} \tag{61}$$

$$\varphi(\ell\omega_0) = \int dt\, \varphi(t) e^{i\ell\omega_0 t} \tag{62}$$

The space and time Fourier transforms are normalized in the usual way by setting the (infinite) intervals of integration to unity when they appear as factors; e.g., a complete statement of (62) could have the form

$$\varphi(\ell\omega_0) = T^{-1} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt\, \varphi(t) e^{i\ell\omega_0 t}$$

where the values of T are integer multiples of $\frac{2\pi}{\omega_0}$, the system has period T, and $\lim_{T\to\infty}$ is implied.

Because our system may be in fact not stationary but have a time variation--e.g., heating caused by the collision term--which is slow compared with the relaxation times of the system, the frequency-dependent coefficients in Eqs. (59) through (62) may have a residual time dependence.

Important properties are

$$\left[\int dt' f_1(tt') f_2(t't'')\right](\omega - n\omega_0, m\omega_0 - \omega) = \sum_{\ell} f_1(\omega - n\omega_0, \ell\omega_0 - \omega) f_2(\omega - \ell\omega_0, m\omega_0 - \omega), \tag{63}$$

$$\left[\int dt' f(tt') \varphi(t')\right](n\omega_0) = \sum_{\ell} f(n\omega_0, -\ell\omega_0) \varphi(\ell\omega_0). \tag{64}$$

A Fourier transform of (35) with the use of (63) is

$$[\delta(t-t')](\omega-n\omega_0, m\omega_0-\omega) = \delta_{mn} = \sum_{\ell} \epsilon_{\underline{k}}^{-1}(\omega-n\omega_0, \ell\omega_0-\omega)\,\epsilon_{\underline{k}}(\omega-\ell\omega_0, m\omega_0-\omega) \tag{65}$$

In terms of an $\underline{\underline{M}}$ defined by

$$M_{mn}(\underline{k},\omega) = \epsilon_{\underline{k}}(\omega-m\omega_0, n\omega_0-\omega) = \delta_{mn} + \chi_{\underline{k}}(\omega-m\omega_0, n\omega_0-\omega) \tag{66}$$

we see that

$$\epsilon_{\underline{k}}^{-1}(\omega-m\omega_0, n\omega_0-\omega) = M_{mn}^{-1}(\underline{k},\omega), \tag{67}$$

which provides most of the argument for existence and uniqueness. An expression forthe nonlinear susceptibility in frequency space follows from using the identity

$$e^{iz\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(z)\, e^{in\theta} \tag{68}$$

in (58):

$$ik\chi_{\alpha\underline{k}}(\omega-n\omega_0, m\omega_0-\omega) = \int d^3v\, \frac{\omega_{p\alpha}^2}{n_\alpha}\hat{k}\cdot\frac{\partial f_\alpha(v,t_0)}{\partial \underline{v}} \sum_p \frac{i}{\omega+p\omega_0-ik\cdot v} J_{m+p}(\underline{k}\cdot\underline{d}_\alpha) J_{n+p}(\underline{k}\cdot\underline{d}_\alpha) \tag{69}$$

A linear susceptibility can be defined by

$$\chi_{\alpha\underline{k}}(\omega) = \frac{\omega_{p\alpha}^2}{kn_\alpha}\int d^3v\, \hat{k}\cdot\frac{\partial f_\alpha(v,t_0)}{\partial \underline{v}}\,\frac{1}{\omega-\underline{k}\cdot\underline{v}+i\epsilon} \tag{70}$$

and in terms of it,

$$\chi_{\alpha\underline{k}}(\omega-n\omega_0, m\omega_0-\omega) = \sum_p \chi_{\alpha k}(\omega+p\omega_0)\, J_{m+p}\, J_{n+p}. \tag{71}$$

The same symbol $\chi_{\underline{k}}$ is used for the linear and nonlinear susceptibilities; they are distinguished by the number of arguments.

Making appropriate substitutions leads to the following expressions, which will be used in Chapter III:

$$
\begin{aligned}
&\langle \delta\rho_{\alpha\underline{k}}(t)\, \delta\vec{E}_{-\underline{k}}(t)\rangle (\ell\omega_0) \\
&\quad = 2\underline{k}i \sum_{m,r} \int \frac{d\omega}{2\pi} \Big\{ \frac{\theta_\alpha \operatorname{Im} \chi_{\alpha k}(\omega)}{\omega} M^{-1}_{\ell+r,m}(\omega,k)^* \\
&\qquad \times J_r(\underline{k}\cdot\underline{d}_\alpha) J_m(\underline{k}\cdot\underline{d}_\alpha) \quad + \\
&\qquad + \chi_{\alpha\underline{k}}(\omega-[\ell+r]\omega_0, m\omega_0-\omega)^* \sum_{n,p} \sum_{\beta=e,i} M^{-1}_{rn}(\omega,\underline{k}) \\
&\qquad \times M^{-1}_{mp}(\omega,k)^* J_n(\underline{k}\cdot\underline{d}_\beta) J_p(\underline{k}\cdot\underline{d}_\beta) \frac{\theta_\beta \operatorname{Im} \chi_{\beta\underline{k}}(\omega)}{\omega} \Big\}
\end{aligned}
\tag{72}
$$

$$
\begin{aligned}
&\langle \delta\vec{j}_{\alpha\underline{k}}(t)\cdot \delta\vec{E}_{-\underline{k}}(t)\rangle (\ell\omega_0) \\
&\quad = 2i \sum_{m,r} \int \frac{d\omega}{2\pi} \Big\{ \frac{\theta_{\alpha'} \operatorname{Im} \chi_{\alpha' k}(\omega)}{\omega} (\omega-[r+\ell]\omega_0) \\
&\qquad \times \chi_{\alpha\underline{k}}(\omega-[r+\ell]\omega_0, m\omega_0-\omega)^* \sum_{n,p} M^{-1}_{rn}(\underline{k},\omega) M^{-1}_{mp}(\underline{k},\omega)^* \\
&\qquad \times J_n(\underline{k}\cdot\underline{d}_{\alpha'}) J_p(\underline{k}\cdot\underline{d}_{\alpha'}) \\
&\qquad + \frac{\theta_\alpha \operatorname{Im} \chi_{\alpha k}(\omega)}{\omega} \Big[(\omega-[\ell+r]\omega_0) \chi_{\alpha\underline{k}}(\omega-[\ell+r]\omega_0, m\omega_0-\omega)^* \\
&\qquad \times \sum_{n,p} M^{-1}_{rn}(\underline{k},\omega) M^{-1}_{mp}(\underline{k},\omega)^* J_n(\underline{k}\cdot\underline{d}_\alpha) J_p(\underline{k}\cdot\underline{d}_\alpha) \\
&\qquad + (\omega-r\omega_0) M^{-1}_{\ell+r,m}(\underline{k},\omega)^* J_r(\underline{k}\cdot\underline{d}_\alpha) J_m(\underline{k}\cdot\underline{d}_\alpha)
\end{aligned}
\tag{73}
$$

The symbol α' labels the species different from α.

5. Collision Term in the Time Domain

Klimontovich and Puchkin (1974), using a similar approach and the same specialization to Eq. (52), derive for Eq. (29) an expression which they give as

$$\overline{I}_{ei} = 2c^2 e_i^2 n_i \frac{\partial}{\partial P_r} \int d^3k \, \frac{k_r k_s}{k^4} \sum_n J_n^2(\underline{k}\cdot\underline{d}_c) \frac{\delta(n\omega_o - \underline{k}\cdot\underline{U})}{|\epsilon(n\omega_o, k)|^2} \frac{\partial F_e}{\partial P_s} \qquad (74)$$

In this, a delta function has been used for the ion distribution function. The variable $\underline{P}$ is the deviation of particle momentum from particle drift momentum, and the remaining time dependence has been averaged over a period of the driving field.

CHAPTER III

HEATING OF ELECTRONS AND IONS BY A HIGH-FREQUENCY FIELD

Fluctuat nec mergitur.

Albert Messiah[1]

In this chapter, the physical system chosen for illustration is the simplified one of a homogeneous two-component plasma driven by an alternating electric field with no magnetic field. It is shown that the energy absorbed from the field by one species can be expressed as the sum of two terms, one representing resistive heating and the other representing an exchange with the other species. The terms are evaluated.

1. Introduction

<u>A hydrodynamic equation</u>

The hydrodynamic equations are the velocity moments of the kinetic equation (II.1). The following exact expression is easily derived, as shown by Chapman and Cowling (1939, page 49, Eq. 3.13,$_2$):

$$I\left\{\phi(\vec{v})\right\} = \frac{\partial}{\partial t} n\overline{\phi} + u_i \frac{\partial}{\partial x_i} n\overline{\phi} + n\overline{\phi}\,\nabla\cdot\underline{u} + \nabla\cdot n\overline{\phi\underline{v}}$$
$$-n\left\{\left[a_i - \frac{\partial u_i}{\partial t} - u_j\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\right]\overline{\frac{\partial\phi}{\partial v_i}} - \frac{\partial u_i}{\partial x_j} v_j \frac{\partial\phi}{\partial v_j}\right\} \quad (1)$$

This equation has been simplified by taking ϕ to be a function of $\underline{v}$

[1]Messiah 1962, front plate for Volume II.

only. The overbar labels the average with respect to the distribution function.

$$I\left\{\phi(\vec{V})\right\} \equiv \int d^3v\, \phi(\underline{V})\, I(\vec{x},\vec{v},t) \tag{2}$$

$I(\vec{x},\vec{v},t)$ is the quantity which customarily appears on the right-hand side of the kinetic equation, the "collision term," as in (II.1) and (II.2) of the preceding chapter.

Deviation of velocity from drift velocity:

$$\underline{V} = \underline{v} - \underline{u}(\underline{x},t) \tag{3}$$

Drift velocity:

$$\underline{u}(\underline{x},t) \equiv n^{-1} \int d^3v\, \underline{v}\, f(\vec{x},\vec{v},t) \tag{4}$$

This definition of $\underline{u}(x,t)$ makes it the actual drift speed, which can differ from the $\underline{u} \equiv \int \underline{a}\, dt$ used elsewhere in this thesis because of the collision term and possibly an initial value. The difference will be important in a more general computation of transport rates; in this thesis, the difference is ignored.

If there are no space gradients, and if ϕ is taken as the particle energy $\frac{1}{2}mV^2$, then (1) reduces to

$$I\left\{\tfrac{1}{2}mV^2\right\} = \frac{\partial}{\partial t}\, \frac{3}{2} n\theta. \tag{5}$$

<u>The separation of terms</u>

In the absence of the driving field, if the system were to begin with nonthermal distribution functions, the electrons would become thermal in a time described by an effective electron-electron relaxation time, then the protons would relax in a time greater by

the root mass ratio, and finally the temperature difference between the two species would relax in a time greater by another root mass ratio (Spitzer 1962, p. 136; Delcroix 1960, p. 113). The interspecies equilibration rate is given by (I.1) with the relaxation time (III.49):

$$\frac{d\theta_e}{dt} = \frac{\theta_i - \theta_e}{\tau_{ei}} \tag{I.1}$$

If a weak driving field is imposed, the species move in opposition, and collisions convert some of the energy of oscillation into thermal energy. Because each kind of particle receives the same impulse in a two-particle encounter, the increment of randomized energy between the two species differs by the mass ratio, so that the ion population heats more slowly than the electron population by the mass ratio. In a weak field with $\theta_e = \theta_i$, the thermal energy of each species grows at the rate

$$\frac{3}{2} n_\alpha \dot{\theta}_\alpha = \frac{1}{2} \mathcal{R}e\, \sigma_\alpha E_0^2 \tag{6}$$

where $\mathcal{R}e\, \sigma_\alpha$, the dissipative part of the conductivity, is given by

$$\mathcal{R}e\, \sigma_\alpha = \frac{ne^2}{m_\alpha \omega_0^2} \nu_0, \tag{7}$$

where ν_0 is the electron-ion collision frequency (I.8), and α is e or i.

The concepts of interspecies energy exchange and resistive heating can still be given meaning in a system with a temperature difference and a strong driving field. The rate at which a species gains energy can be derived from the hydrodynamic equation (1).

If the Klimontovich collision term (II.6) is used in (2) for the left-hand side of (5), then (5) is

$$\frac{\partial}{\partial t}\ \frac{3}{2} n\theta = \int d^3v \langle \delta a \delta f \rangle \cdot \frac{\partial \frac{1}{2} m v^2}{\partial \underline{v}} \tag{8}$$

$$= \int d^3v \langle q \delta \underline{E} \cdot (\underline{v} - \underline{u}) \delta f \rangle \tag{9}$$

$$\frac{3}{2} n_\alpha \frac{\partial \theta_\alpha}{\partial t} = \langle \delta \underline{E} \cdot \delta \underline{j}_\alpha \rangle - \underline{u}_\alpha \cdot \langle \delta \underline{E} \delta \rho_\alpha \rangle \tag{10}$$

by appropriate definition of $\delta \vec{j}$ and $\delta \rho$, and with species subscripts included for the last line.

We will find that the second term on the right-hand side is quadratic in field strength for weak fields, so that it can plausibly be defined as the resistive heating. In the absence of a field, the first term is given by (I.1), so it can be defined as a heat exchange term These definitions aren't unique; e.g., any quantity $(\theta_\alpha - \theta_{\alpha'})E^2 f(t)$ could be added to one term and subtracted from the other without destroying the properties which motivate those labels.

Average over a period of the driving field

Each of the bracketed quantities in (10) has an expansion of the form (II.61). Taking the time averages gives

$$\overline{\langle \delta \underline{E} \cdot \delta \underline{j}_\alpha \rangle} = \langle \delta \underline{E} \cdot \delta \underline{j}_\alpha \rangle_{\ell=0} \tag{11}$$

and the heating rate

$$\frac{1}{2} \sigma_D E_0^2 \equiv \overline{-\underline{u}_\alpha \cdot \langle \delta \underline{E} \delta \rho_\alpha \rangle} = \underline{u}_{0\alpha} \cos \omega_0 t \left[e^{-i\omega_0 t} \langle \rangle_1 + e^{i\omega_0 t} \langle \rangle_{-1} \right]$$

$$= \mathrm{Re}\ \omega_0 \underline{d}_\alpha \cdot \langle \delta \underline{E} \delta \rho_\alpha \rangle_{\ell=1}. \tag{12}$$

This defines the dissipative part of the conductivity $\sigma_D = \mathrm{Re}\, \sigma$.

Matrix notation for the frequency dependence

The expressions (II.72) and (II.73) for the resistive and equilibration terms are given a cleaner appearance with more compact notation. With arguments $\underline{k}$ and ω understood, we can write

$$K^{\alpha}_{mn} = \chi_{\alpha k}(\omega - m\omega_0, n\omega_0 - \omega) \tag{13}$$

$$\underline{\underline{M}} = \underline{\underline{1}} + \underline{\underline{K}}^{e} + \underline{\underline{K}}^{i} \tag{14}$$

$$N = M^{-1} \tag{15}$$

$$\Omega_{mn} = (\omega - n\omega_0)\,\delta_{mn} \tag{16}$$

$$\underline{J} = \underline{J}^{e} = col\left\{J_n(\underline{k}\cdot\underline{d}_e)\right\} \tag{17}$$

$$\underline{L} = \underline{J}^{i} = col\left\{J_n(\underline{k}\cdot\underline{d}_i)\right\} \tag{18}$$

The underlining will sometimes be omitted from the matrix and vector symbols. We define a shift operator by

$$S_{mn} = \delta_{m+1,n} \tag{19}$$

so that it has the property

$$(SZ)_{mn} = Z_{m+1,n}. \tag{20}$$

Clearly true is

$$\underline{J}^{+}\underline{S}^{n}\underline{J} = \delta_{n,0} \tag{21}$$

in which $\underline{J}^{+}$ is the Hermitian conjugate of $\underline{J}$.

Remembering that the linear susceptibility depends only on the magnitude of $\underline{k}$, we get from (II.71)

$$K^{e}_{mn} = \sum_{\ell} \chi_{ek}(\omega + \ell\omega_0)\, J_{\ell+m} J_{\ell+n}. \tag{22}$$

Effect of small mass ratio

The argument of the ion Bessel function is $\underline{k}\cdot\underline{d}_i = k\mu d_e\sqrt{\frac{m_e}{m_i}}$. We will take

$$L_m = \delta_{m,0} \tag{23}$$

which is equivalent to ignoring terms of order $\underline{k}\cdot\underline{d}_i$. Thus we have assumed the inequality

$$1 >> k\mu d_e\sqrt{\frac{m_e}{m_i}} = k\mu\frac{\omega_p}{\omega_0 k_0}2E\sqrt{\frac{m_e}{m_i}} \tag{24}$$

in which the dimensionless field strength (I.29) appears. So even though our result is intended to apply for $k_D << k$ and $E > 1$, we suppose the mass ratio to be strong enough to enforce (24). That condition imposes an upper limit to to the value of k allowed when the sum is done over $\underline{k}$:

$$k < \frac{\omega_0}{\omega_p}\frac{k_0}{E}\sqrt{\frac{m_i}{m_e}} \tag{25}$$

We notice also that the quantity $\mathrm{Im}\,\chi_i\,(k,\omega)$ has an exponential factor with argument

$$\left(\frac{\omega}{k v_i\sqrt{2}}\right)^2 = \left(\frac{\omega}{\omega_p}\frac{k_0}{k\sqrt{2}}\right)^2\frac{m_i}{m_e} \tag{26}$$

The condition for that to be large when ω is near ω_0 is

$$1 << \frac{\omega_0^2 k_0^2}{2\omega_p^2 k^2}\frac{m_i}{m_e}. \tag{27}$$

But (27) is already implied by (24), because we allow $E>1$.

A consequence of (23) and (II.71) is

$$K^i_{mn} = \delta_{mn} \chi_{i\underline{k}}(\omega + m\omega_0) \qquad \text{(no sum).} \tag{28}$$

2. Resistive Heating

Earlier work

The rate at which a plasma absorbs energy from a weak high-frequency electric field was calculated by Dawson and Oberman (1962; Oberman, Ron, and Dawson 1962; Dawson and Oberman 1963). Rand (1964) derived a single-particle absorptivity but didn't average over the electron distribution function. Silin (1965b) calculated the nonlinear absorptivity and reached an expression equivalent to (III.41). Kaw and Salat (1968) extended the linear result to stronger fields by taking more terms in the expansion of Bessel functions with arguments proportional to field strength. Kidder (1971) reaches a simple expression for the absorption of a circularly polarized wave. A quantum-mechanical derivation by Seely and Harris (1973) used an erroneous approximation and quasi-physical arguments for the evaluation of an integral, and reached a conductivity which lacked the Coulomb logarithm (McKinnis and Goldman 1975). Klimontovich and Puchkin (1974) give a result similar to that of Silin.

Simplified expression

Equation (II.72) in the compact notation is, because only the real part contributes,

$$\mathrm{Re}\,\langle \delta \underline{E}\, \delta \rho_\alpha \rangle_{\underline{k},\ell=1} = -\underline{k} \int \frac{d\omega}{\pi} \Big\{ \frac{\theta_\alpha \mathrm{Im}\,\chi_\alpha(k,\omega)}{\omega} \mathrm{Im}\, J_\alpha^+ N^+ S^+ J_\alpha$$

$$+ \sum_\beta \frac{\theta_\beta \mathrm{Im}\,\chi_\beta(k,\omega)}{\omega} \mathrm{Im}\, J_\beta^+ N^+ K_\alpha^+ S^+ N J_\beta \Big\} \quad (29)$$

$$= -\underline{k} \int \frac{d\omega}{\pi} \Big\{ \frac{\theta_\alpha \mathrm{Im}\,\chi_\alpha}{\omega} \mathrm{Im} \big[J_\alpha^+ N^+ S^+ J_\alpha + J_\alpha^+ N^+ K_\alpha^+ S^+ N J_\alpha$$

$$+ \frac{\theta_{\alpha'} \mathrm{Im}\,\chi_{\alpha'}}{\omega} \mathrm{Im}\, J_{\alpha'}^+ N^+ K_\alpha^+ S^+ N J_{\alpha'} \Big\} \quad (30)$$

$$\mathrm{Re}\,\langle \delta \underline{E}\, \delta \rho_e \rangle_{\underline{k},\ell=1} = \underline{k} \int \frac{d\omega}{\pi} \Big\{ \frac{\theta_e \mathrm{Im}\,\chi_e}{\omega} \mathrm{Im}\, J^+ N^+ \big[M^+ S + S K_e \big] N J$$

$$- \frac{\theta_i \mathrm{Im}\,\chi_i}{\omega} \mathrm{Im}\, L^+ N^+ S K_e N L \Big\} \quad (31)$$

Expression for unrestricted k

This expression can be developed further, but we will use only the large-k form. Silin (1965b) and Klimontovich and Puchkov (1974) give expressions which are similar to the one which would follow from (31).

Large-wave-number form

The susceptibilities are products of k_D^2/k^2 with bounded quantities, so that both M and N become unity at large wave numbers. The fractions in (31) multiply terms which are of comparable

size, so the ion term dominates. The contributing term is

$$L^{+} S K_e L = K^{e}_{10} \tag{32}$$

$$= \sum_{\ell} J_{\ell} J_{\ell-1} \chi_e(\omega + \ell\omega_0) \tag{33}$$

$$Re \langle \delta E \delta \rho_e \rangle_{\underline{k}, \ell=1} = -k \int \frac{d\omega}{\pi} \sum_{\ell} \frac{\theta_i Im\, \chi_i}{\omega} J_{\ell} J_{\ell-1} Im\, \chi_e(\omega + \ell\omega_0)$$

$$= k \int \frac{d\omega}{\pi} \frac{\theta_i\, Im\, \chi_i}{2\omega} \sum_{\ell} J_{\ell} J_{\ell-1}\, Im\left[\chi_e(\omega - \ell\omega_0) - \chi_e(\omega + \ell\omega_0)\right] \tag{34}$$

The term in brackets is an odd function of ℓ, say A_{ℓ}. The sum is

$$\sum_{\ell} J_{\ell-1} J_{\ell} A_{\ell} = -\sum_{\ell} J_{\ell} J_{\ell-1} A_{\ell}$$

by changing the sign of ℓ, and also

$$= \sum_{\ell} -J_{\ell} J_{\ell+1} A_{\ell} \tag{35}$$

from the symmetry of the Bessel function.

$$\therefore = -\frac{1}{2} \sum A_{\ell} (J_{\ell} J_{\ell+1} + J_{\ell} J_{\ell-1})$$

$$= -\sum_{\ell} \frac{\ell}{\underline{k} \cdot \underline{d}_e} A_{\ell} J_{\ell}^2 \tag{36}$$

From (12), the absorptivity is

$$\sigma_D = \frac{2}{E_0^2} \omega_0 \sum_{\underline{k}} \hat{k} \cdot \underline{d}_e\, k \int \frac{d\omega}{\pi} \frac{\theta_i\, Im\, \chi_i}{\omega} \sum_{\ell} \frac{\ell}{\underline{k} \cdot \underline{d}_e} J_{\ell}^2\, Im\, \chi_e(\omega + \ell\omega_0) \tag{37}$$

The ω argument in χ_e doesn't contribute much to the integral, so

$$\sigma_D = \frac{\omega_0}{2E^2} \left(\frac{e}{m\omega_0 v_e}\right)^2 \sum_{\underline{k}} \frac{4\pi n e^2}{k^2} Z \sum_{\ell} \ell J_{\ell}^2\, Im\, \chi_e(\ell\omega_0). \tag{38}$$

The sum on ℓ , by (I.34), becomes

$$\frac{k v_e \sqrt{2}}{\omega_0} \frac{k_D^2}{k^2} \pi^{\frac{1}{2}} \Gamma_1(\zeta). \tag{39}$$

The prescription (I.7) gives

$$\sigma_D = \frac{\omega_0 e^2 E^{-2}}{2(m\omega_0 v_e)^2} \frac{m v_e \theta_e 3\nu_0}{ne^4 Z\, 2(2\pi)^{3/2}} \int_0^1 d\mu\, 4\pi n e^2 Z \frac{v_e \sqrt{2}}{\omega_0} k_D^2 \pi^{\frac{1}{2}} \Gamma_1(\zeta) \tag{40}$$

$$= \frac{ne^2}{m\omega_0^2} \nu_0\, 3\, E^{-3} \int_0^E d\zeta\, \Gamma_1(\zeta). \tag{41}$$

Inverse Bremsstrahlung rate in a weak field:

A check on the compuatation is the value of σ_D in weak fields:

$$\Gamma_1(\zeta) = \zeta^2 \left[1 - \tfrac{3}{2}\zeta^2 + \cdots \right] \tag{42}$$

$$\sigma_D(0) = \frac{ne^2}{m\omega_0^2} \nu_0. \tag{43}$$

Asymptotic form in a strong field:

If E is large, we can use the asymptotic form (I.40):

$$\Gamma_1(\zeta) \sim \frac{1}{2\sqrt{2\pi}} \zeta^{-1} \left\{ 1 + \tfrac{3}{8}\zeta^{-2} + \cdots \right\} \tag{44}$$

$$\sigma_D \sim \left(\int_0^{\mathcal{O}(1)} + \int_{\mathcal{O}(1)}^{E} \right) dE \longrightarrow \int_{\mathcal{O}(1)}^{E} dE \tag{45}$$

$$\sigma_D \sim \frac{ne^2}{m\omega_0^2} \nu_0 3 E^{-3} \int^E d\zeta\, \zeta^{-1}/2\sqrt{2\pi}$$

$$\sim \frac{ne^2}{m\omega_0^2} \nu_0 \frac{3}{2\sqrt{2\pi}} E^{-3} \ln E \tag{46}$$

The electron heating rate becomes

$$\tfrac{1}{2} \sigma_D E_0^2 \sim \frac{3\nu_0}{\sqrt{2\pi}} n\theta_e E^{-1} \ln E \tag{47}$$

Numerical values

Fig. 3 and Table 1 show values of the dissipative part of the conductivity in units of $\frac{ne^2}{m\omega^2}\nu_o$. It should be noticed that ν_o itself depends on E through the Coulomb logarithm.

In the measurement by Brownell et al. (1974 and 1975), a value is seen of $\sigma_D(.39) = 0.42$, differing from the Salat and Kaw value 0.78. This computation gives $\sigma_D(.39) = 0.87$, which differs even more from the experimental value.

E	SIGMA
0.00	1.0000
.05	.9977
.10	.9911
.15	.9800
.20	.9648
.25	.9458
.30	.9232
.35	.8974
.40	.8688
.45	.8378
.50	.8050
.55	.7707
.60	.7354
.65	.6996
.70	.6635
.75	.6276
.80	.5922
.85	.5575
.90	.5239
.95	.4915
1.00	.4604
1.50	.2328
2.00	.1242
2.50	.0730
3.00	.0465
3.50	.0315
4.00	.0224
4.50	.0165
5.00	.0126

Table 1. Numerical values to accompany Fig. 3.

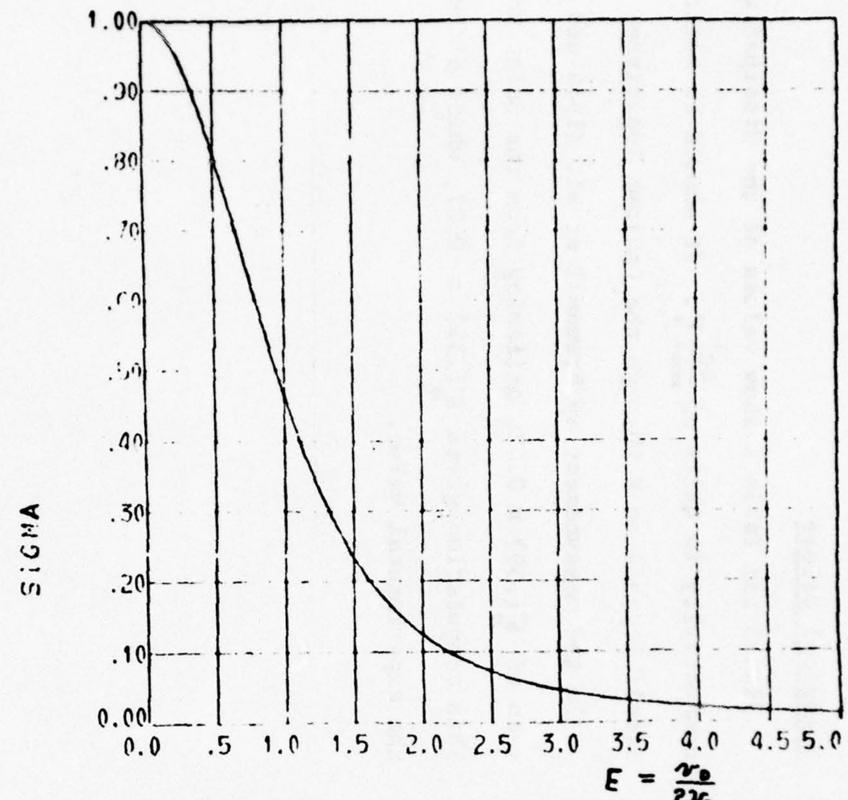

Fig. 3. Real (dissipative) part of the AC conductivity in units of its zero-field value.

Earlier work

For a system without a driving field, Landau (1936) gives for the relaxation rate in (I.1), page 3, the value

$$\tau_{ei}^{-1} = 2\nu_0 \frac{m}{M}. \tag{48}$$

Spitzer (1940), discussing the passage of a star through a globular cluster, gives

$$\tau_{ei}^{-1} = 2\nu_0 \frac{m}{M} \left(1 + \frac{m}{M} \frac{\theta_i}{\theta_e}\right)^{-3/2} \tag{49}$$

without constraints on mass or temperature ratio.

<u>Simplified expression</u>

In compact notation, (II.73) is

$$\langle \delta \vec{j}_{\alpha \underline{k}} \cdot \delta \underline{E}_{-\underline{k}} \rangle_{\ell=0} = 2i \int \frac{d\omega}{2\pi} \left\{ \frac{\theta_{\alpha'} \operatorname{Im}\chi_{\alpha'}(k,\omega)}{\omega} J^{+}_{\alpha'} N^{+} K^{+}_{\alpha} \Omega N J_{\alpha} + \frac{\theta_{\alpha} \operatorname{Im}\chi_{\alpha}(k,\omega)}{\omega} \left[J^{+}_{\alpha} N^{+} K^{+}_{\alpha} \Omega N J_{\alpha} + J^{+}_{\alpha} N^{+} \Omega J_{\alpha} \right] \right\} \tag{50}$$

$$= -\int \frac{d\omega}{\pi} \left\{ \frac{\theta_{\alpha'} \operatorname{Im}\chi_{\alpha'}}{\omega} \operatorname{Im}(J^{+}_{\alpha'} N^{+} K_{\alpha'} \Omega N J_{\alpha'} + \frac{\theta_{\alpha} \operatorname{Im}\chi_{\alpha}}{\omega} \operatorname{Im} J^{+}_{\alpha} N^{+}_{\alpha} \left[K^{+}_{\alpha} \Omega + M \Omega \right] N J_{\alpha} \right. \tag{51}$$

The contributing part of the bracketed expression is the antihermitian part, which is half of

$$K^{+}_{\alpha} \Omega + \Omega M - \Omega K_{\alpha} - M^{+} \Omega = -(1 + K^{+}_{\alpha'}) \Omega + \Omega (1 + K_{\alpha'}) = \Omega K_{\alpha'} - K^{+}_{\alpha'} \Omega \tag{52}$$

and the simplified form of the general expression is

$$\langle \delta \vec{j}_{\alpha} \cdot \delta \underline{E} \rangle_{\underline{k},0} = \int \frac{d\omega}{\pi} \left\{ \frac{\theta_{\alpha'} \operatorname{Im}\chi_{\alpha'}}{\omega} \operatorname{Im} J^{+}_{\alpha'} N^{+} \Omega K_{\alpha} N J_{\alpha} - \frac{\theta_{\alpha} \operatorname{Im}\chi_{\alpha}}{\omega} \operatorname{Im} J^{+}_{\alpha} N^{+} \Omega K_{\alpha'} N J_{\alpha} \right\} \tag{53}$$

The symmetric form makes obvious the property which makes this the "equipartition" term:

$$\left\langle \delta \vec{j}_{e} \cdot \delta \underline{E} \right\rangle_{\underline{k},0} + \left\langle \delta \vec{j}_{i} \cdot \delta \underline{E} \right\rangle_{\underline{k},0} = 0 \tag{54}$$

The energy gained by the ions is lost by the electrons.

Large-wave-number form

The susceptibilities $\chi_s(\omega)$ are (k_D^2/k^2) times quantities which are bounded above, so that M and N are unit matrices at large k.

$$\langle \delta \vec{J}_e \cdot \delta \underline{E} \rangle_{\underline{k}, l=0} = \int \frac{d\omega}{\pi} \left\{ \frac{\theta_i \, Im\, \chi_i}{\omega} Im\, L^{+} \Omega K_e L - \frac{\theta_e \, Im\, \chi_e}{\omega} Im\, \overset{+}{J} \Omega K_i J \right\} \quad (55)$$

Reduction of terms:

$$L^{+} \Omega K_e L = (\Omega K_e)_{00} = \Omega_{00} K^e_{00} = \omega K^e_{\infty} = \omega \sum_n J_n^2 \chi_e(\omega + n\omega_0) \quad (56)$$

$$\frac{\theta_e \, Im\, \chi_e}{\omega} Im\, J_n \Omega_{np} K^i_{pq} J_q = \frac{\theta_e \, Im\, \chi_e(\omega)}{\omega} \sum_n (\omega - n\omega_0) J_n^2 \, Im\, \chi_i(\omega - n\omega_0)$$

$$= \sum_n \frac{\theta_e \, Im\, \chi_e(\omega + n\omega_0)}{\omega + n\omega_0} \omega J_n^2 \, Im\, \chi_i(\omega) \quad (57)$$

The rate of heat transfer from ions to electrons is

$$\langle \delta \vec{J}_e \cdot \delta \underline{E} \rangle_{\underline{k}, 0} = \int \frac{d\omega}{\pi} \left\{ \frac{\theta_i \, Im\, \chi_i(\omega)}{\omega} Im\, \omega \sum_n J_n^2 \chi_e(\omega + n\omega_0) \right.$$

$$\left. - \sum_n \omega J_n^2 \, Im\, \chi_i(\omega) \frac{\theta_e \, Im\, \chi_e(\omega + n\omega_0)}{\omega + n\omega_0} \right\} \quad (58)$$

$$= \int \frac{d\omega}{\pi} Im\, \chi_i(\omega) \sum_n J_n^2 \frac{Im\, \chi_e(\omega + n\omega_0)}{\omega + n\omega_0} \left\{ \theta_i(\omega + n\omega_0) - \theta_e \omega \right\} \quad (59)$$

Because $\dfrac{\chi_e(\omega + n\omega_0)}{\omega + n\omega_0}$ is an even function of its argument, and $\dfrac{Im\, \chi_i(\omega)}{\omega}$ is almost a delta function as a multiplier of it, we can put

$$\frac{\chi_e(\omega + n\omega_0)}{\omega + n\omega_0} \pm \frac{\chi_e(\omega - n\omega_0)}{\omega - n\omega_0} \approx \begin{bmatrix} 2 \dfrac{\chi_e(n\omega_0)}{n\omega_0} \\ 2\omega \dfrac{d}{d n\omega_0} \dfrac{\chi_e(n\omega_0)}{n\omega_0} \end{bmatrix}$$

$$\approx 2 \frac{\chi_e(n\omega_0)}{n\omega_0} \begin{bmatrix} 1 \\ - n\omega_0 \omega / (k v_e)^2 \end{bmatrix} \quad (60)$$

where the top quantity in brackets corresponds to the top sign.

$$\langle \delta \vec{j}_e \cdot \delta \underline{E} \rangle_{\underline{k}, \ell=0} \approx \int \frac{d\omega}{\pi} \operatorname{Im}\chi_i(\omega) \sum_n J_n^2 \frac{\operatorname{Im}\chi_e(n\omega_0)}{n\omega_0} \left\{ \omega(\theta_i - \theta_e) - n\omega_0 \theta_i \frac{n\omega_0 \omega}{(k v_e)^2} \right\} \quad (61)$$

$$\langle \delta \vec{J}_e \cdot \delta \underline{E} \rangle_{\underline{k}} \approx \frac{m_e}{m_i} \omega_p^2 \sum_n J_n^2 \frac{\operatorname{Im}\chi_e(n\omega_0)}{n\omega_0} \left\{ \theta_i - \theta_e - 2\theta \left(\frac{n\omega_0}{k v_e \sqrt{2}} \right)^2 \right\} \quad (62)$$

Applying the k limit (I.34) and then the $\underline{k}$ sum (I.7) gives

$$\langle \delta \vec{J}_e \cdot \delta \underline{E} \rangle = \sum_{\underline{k}} \frac{m_e}{m_i} Z \omega_{pe}^2 \frac{1}{k v_e \sqrt{2}} \left\{ \frac{k_D^2}{k^2} \pi^{\frac{1}{2}} \right\} \left\{ (\theta_i - \theta_e) \Gamma_0(\zeta) - 2\theta_i \Gamma_1(\zeta) \right\} \quad (63)$$

$$= \frac{m_e}{m_i} Z \frac{\omega_{pe}^2}{k v_e \sqrt{2}} \frac{k_D^2}{k^2} \pi^{\frac{1}{2}} \frac{m v_e \theta_e}{n e^4 Z} \frac{3\nu_0}{2(2\pi)^{\frac{1}{2}}} \int_0^1 d\mu \left\{ (\theta_i - \theta_e) \Gamma_0 - 2\theta_i \Gamma_1 \right\}$$

$$= \frac{m_e}{m_i} 3 n \nu_0 \int_0^1 d\mu \left[(\theta_i - \theta_e) \Gamma_0(\zeta) - 2\theta_i \Gamma_1(\zeta) \right]. \quad (64)$$

<u>Relaxation rate in vanishing field:</u>

When E = 0, $\Gamma_0(\mu E) = 1$ and $\Gamma_1(\mu E) = 0$, so that

$$\langle \delta \vec{J}_e \cdot \delta \underline{E} \rangle = \frac{m_e}{m_i} 3 n \nu_0 (\theta_i - \theta_e), \quad (65)$$

which agrees with (45).

<u>Asymptotic form in a strong field:</u>

The asymptotic form follows from (I.40):

$$\Gamma_0 \sim 2(2\pi)^{-\frac{1}{2}} \zeta^{-1} \qquad \Gamma_1 \sim (2\pi)^{-\frac{1}{2}} \zeta^{-1}$$

$$\langle \delta \vec{J}_e \cdot \delta \underline{E} \rangle \sim -\frac{m_e}{m_i} n \nu_0 \, 6 (2\pi)^{-\frac{1}{2}} \theta_e E^{-1} \ln E$$

Numerical values

For tabulation, (64) can be written as

$$\langle S_{\underline{I}} \cdot S_{\underline{E}} \rangle = 3n\frac{m}{M}\nu_\theta \left\{ \theta_i - \theta_e - \theta_i \mathcal{I}(E) \right\} \tag{67}$$

where ν_θ(E) and $\mathcal{I}$(E) are defined by equivalence of the expression to (64). The equilibration rate for several values of the electron-ion temperature ratio is shown in Fig. 4 and Table 3; as for all these functions, it must be remembered that there is an additional dependence on field strength through the Coulomb logarithm.

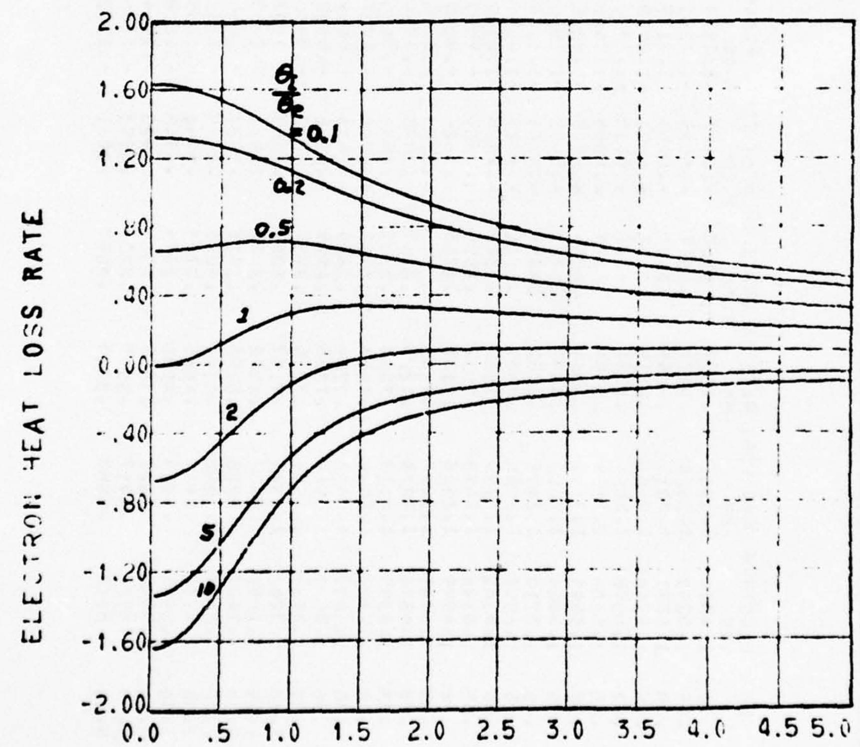

Fig. 4. Electron heat loss rate in units of $3n\frac{m}{M}\nu_0\frac{1}{2}(\theta_i+\theta_e)$, for several values of θ_i/θ_e .

E	ELECTRON HEAT LOSS RATE IN UNITS OF MEAN TEMP. PARAMETER TI/TE .10	.20	.50	1.00	2.00	5.00	10.00	THERMNU	UPSILON
0.00	1.6364	1.3333	.6667	-0.0000	-.6667	-1.3333	-1.6364	1.0000	0.0000
.05	1.6353	1.3328	.6672	.0017	-.6639	-1.3295	-1.6320	.9992	.0017
.10	1.6321	1.3311	.6689	.0066	-.6556	-1.3179	-1.6189	.9967	.0066
.15	1.6269	1.3283	.6715	.0147	-.6421	-1.2989	-1.5975	.9926	.0148
.20	1.6196	1.3244	.6751	.0257	-.6236	-1.2730	-1.5682	.9869	.0261
.25	1.6104	1.3195	.6794	.0394	-.6006	-1.2406	-1.5316	.9797	.0402
.30	1.5993	1.3134	.6844	.0554	-.5736	-1.2026	-1.4885	.9712	.0570
.35	1.5864	1.3062	.6897	.0733	-.5432	-1.1596	-1.4398	.9613	.0762
.40	1.5718	1.2979	.6953	.0927	-.5100	-1.1126	-1.3865	.9503	.0975
.45	1.5558	1.2886	.7009	.1131	-.4746	-1.0624	-1.3296	.9382	.1206
.50	1.5383	1.2783	.7062	.1342	-.4379	-1.0100	-1.2700	.9252	.1450
.55	1.5196	1.2670	.7112	.1554	-.4004	-.9562	-1.2088	.9114	.1705
.60	1.4999	1.2548	.7157	.1765	-.3626	-.9018	-1.1469	.8970	.1968
.65	1.4791	1.2417	.7194	.1970	-.3253	-.8476	-1.0851	.8820	.2234
.70	1.4576	1.2278	.7223	.2167	-.2888	-.7944	-1.0242	.8667	.2501
.75	1.4355	1.2133	.7243	.2353	-.2536	-.7426	-.9648	.8511	.2765
.80	1.4129	1.1980	.7253	.2527	-.2200	-.6927	-.9075	.8354	.3025
.85	1.3899	1.1823	.7254	.2686	-.1883	-.6451	-.8528	.8196	.3277
.90	1.3667	1.1660	.7245	.2829	-.1586	-.6002	-.8009	.8038	.3520
.95	1.3434	1.1494	.7226	.2957	-.1312	-.5580	-.7520	.7881	.3752
1.00	1.3202	1.1325	.7197	.3069	-.1059	-.5187	-.7063	.7727	.3972
1.50	1.1052	.9652	.6572	.3492	.0412	-.2668	-.4069	.6366	.5485
2.00	.9407	.8278	.5795	.3311	.0827	-.1656	-.2785	.5381	.6153
2.50	.8197	.7242	.5141	.3040	.0939	-.1162	-.2117	.4671	.6508
3.00	.7281	.6449	.4620	.2790	.0960	-.0869	-.1701	.4140	.6740
3.50	.6565	.5826	.4200	.2574	.0948	-.0678	-.1417	.3726	.6908
4.00	.5990	.5323	.3856	.2389	.0923	-.0544	-.1211	.3395	.7038
4.50	.5516	.4907	.3569	.2231	.0892	-.0446	-.1054	.3123	.7143
5.00	.5118	.4558	.3325	.2093	.0861	-.0372	-.0932	.2895	.7230

Table 2. Numerical values corresponding to Fig. 4: Rate of loss of electron heat to ions.

CHAPTER IV

THERMAL CONDUCTIVITY

Ego in mille formas transmutavi, sed operam meam improbum Problema perpetuo lusit.[1]
James Bernoulli

1. Introduction

The usual approach to calculating quantities dependent on small deviations of the distribution function from equilibrium is that of Chapman and Enskog (Chapman and Cowling 1939; Grad 1960). It involves defining a smallness parameter

$$\mu = \frac{\nu_c}{\nu_0}\left|\nabla \ln f(\vec{x},\vec{v},t)\right|, \tag{1}$$

expansion of the distribution functions in powers of μ,

$$f_\alpha = f_{0\alpha}\left(1 + \mu\varphi_\alpha^{(1)} + \mu^2\varphi_\alpha^{(2)} + \ldots\right),$$

and writing the kinetic equation in the form

$$\mathcal{L}_\alpha f_\alpha \equiv \left(\frac{\partial}{\partial t} + \underline{v}\cdot\nabla + \underline{a}\cdot\frac{\partial}{\partial \underline{v}}\right) f_\alpha = \frac{1}{\mu}\sum_\beta I_\alpha(f_\alpha, f_\beta). \tag{2}$$

The method is presented by Chapman and Cowling as an expansion to all orders in μ. The μ^{-1} term requires $\sum_\beta I_\alpha(f_{\alpha 0}, f_{\beta 0})$ to vanish and thus determines $f_{\alpha 0}$ to be locally Maxwellian. The zero-order term is the

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[1] Quoted by Watson (1944), p. 1.

one which describes irreversible processes in a near-equilibrium plasma (deGroot 1951), and in practice is usually the highest one calculated. The collision term is bilinear in f_α and f_β , so

$$\mathcal{L}_\alpha f_{o\alpha}(1+\varphi_\alpha) = \sum_\beta I_\alpha \left[f_{o\alpha}(1+\varphi_\alpha), f_{o\beta}(1+\varphi_\beta)\right] \longrightarrow \sum_\beta I_\alpha[f_{o\alpha}, f_{o\beta}\varphi_\beta] + I_\alpha[f_{o\alpha}\varphi_\alpha, f_{o\beta}], \tag{3}$$

which is a linear integral equation in φ and subject to attack by classical methods. The φ's are expressed as infinite sums of orthogonal polynomials, and an appropriate integration gives a matrix equation which can be inverted to a solution for the coefficients of the polynomials.

The thermal conductivity to be derived in this chapter can be defined as the $\underline{\underline{K}}$ in

$$\underline{S} = -\underline{\underline{K}}(\underline{F}) \cdot \nabla\theta. \tag{4}$$

Such an expression can be expected a priori to apply if μ is small, as we assume here. $\underline{S}$ is the heat flux

$$\vec{S}_e \equiv \int d^3v \, \tfrac{1}{2} m v^2 \underline{v} \, f_e(\vec{x}, \vec{v}, t). \tag{5}$$

It is half the quantity called the "thermal flux" in the book by Klimontovich (1967). A dimensionless form of $\underline{S}$ is convenient:

$$\underline{\mathcal{S}} = \frac{S}{n_e \theta_e v_e} \tag{6}$$

If the "Krook model" value (Boyd and Sanderson 1969) of the conductivity is used as a unit,

$$K_0 = \frac{5}{2}\frac{n\theta}{m\nu_0}, \tag{7}$$

then

$$\underline{S} = -\left(\frac{\underline{\underline{K}}}{K_0}\right)\cdot K_0 \nabla\theta \tag{8}$$

and

$$\mathcal{S} = -\left(\frac{\underline{\underline{K}}}{K_0}\right)\cdot \frac{5}{2}\,\frac{\nu_e}{\theta V_0}\,\nabla\theta. \tag{9}$$

If the only anisotropies of the plasma are the driving field $\underline{E}$ and the temperature gradient, $\underline{\underline{K}}$ will have the form

$$K_{ij} = (\delta_{ij} - \frac{E_i E_j}{E^2})K_\perp + \frac{E_i E_j}{E^2} K_{\parallel}, \tag{10}$$

where $K_\perp$ and $K_\parallel$ play the roles suggested by their labels because it follows directly that

$$\mathcal{S}_\parallel = -\hat{E}\cdot\underline{\underline{K}}\cdot\nabla\theta = -K_\parallel(\hat{E}\cdot\nabla\theta) = -K_\parallel(\nabla\theta)_\parallel$$

and

$$\mathcal{S}_\perp = \mathcal{S} - \hat{E}\mathcal{S}_\parallel = -\underline{\underline{K}}\cdot\nabla\theta + K_\parallel \hat{E}\hat{E}\cdot\nabla\theta = K_\perp(\nabla\theta)_\perp .$$

Because the interaction cross-section for a two-particle collision decreases with the speed, and adding the diffusion velocity to the driving velocity is more effective when the two are parallel, we expect

$$K_\parallel \geq K_\perp .$$

In this chapter, a version of the Chapman-Enskog method is used. The thermal conductivity is computed for a field strength which is

constrained only by nonrelativistic dynamics and motionless ions. We show that the thermal conductivity implied by the collision term of Klimontovich has the behavior as a function of field strength which is sketched in Figure 5.

2. Earlier Work

Landshoff (1949, 1951) calculated the thermal conductivity of an equilibrium plasma, and the other coefficients in Eq.(I.2), with a fifth-order Chapman-Enskog expansion. Spitzer et al. (Spitzer and Härm 1953; Cohen, Spitzer, and Routly 1950) solved numerically a second-order differential equation for a Fokker-Planck collision term for an exact result. After electronic computers became more useful, other workers (Hochstim 1969 and references therein) did large-order matrix inversions with inclusion of dependence on magnetic field.

The accepted expression for thermal conductivity, including allowance for the thermoelectric effect, is given by Spitzer (1962):

$$K = 1.28 \frac{5 n \theta}{2 m \nu_0} = 1.28 K_0 \qquad (11)$$

The conductivity of a plasma driven to superthermal speeds seems not to have been calculated. Because the interactions of particles of one species among themselves aren't affected by the field--in an oscillating frame of reference, they don't see the field--and because the interaction cross-section decreases with increasing relative speed, the thermal flux of a plasma driven strongly enough will be limited by electron-electron collisions. The result of Everett (1963) for strong fields is

$$\frac{K}{K_0} = \frac{5}{2\sqrt{2}} = 1.760. \qquad (12)$$

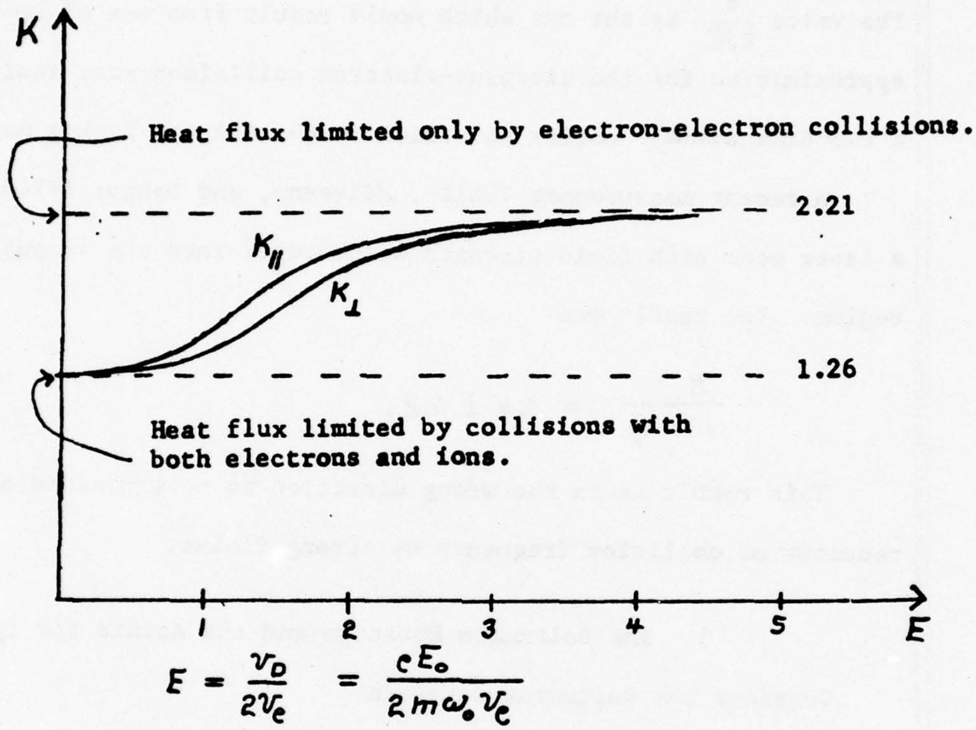

Fig. 5. Sketch describing expected behavior of thermal conductivities perpendicular and parallel to the driving field. The unit for the conductivities $K_{\perp}$ and $K_{\parallel}$ is $\frac{5}{2}\frac{n\theta_e}{mv_0}$. The asymptotes are the values from a 3 x 3 matrix inversion known from Landshoff (1949).

The result by Kelleher and Everett (1967) that purports to be the same seems to be in fact

$$\frac{K}{K_0} = 1.121. \tag{13}$$

The value $\frac{5}{2\sqrt{2}}$ is the one which would result from use of the Grad approximation for the electron-electron collisions--or, equivalently, a two-dimensional matrix inversion in the Chapman-Enskog method.

A recent measurement (White, Kilkenny, and Dangor 1974) involved a laser beam with field strength which edged into the strong-field regime. The result was

$$\frac{K}{1.28\,K_0} = 0.4 \pm 0.2. \tag{14}$$

This result is in the wrong direction to be explained by the reduction of collision frequency by strong fields.

3. The Boltzmann Equation and the Ansatz for f_e

Consider the Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\nabla + \underline{a}\cdot\frac{\partial}{\partial \underline{v}}\right) f(\underline{x},\underline{v},t) = I(\underline{x},\underline{v},t). \tag{15}$$

We can express f(x,v,t) in the form

$$f(\underline{x},\underline{v},t) = v_e^{-3} f_0(\underline{w}) \left\{ 1 + \sum_{\nu=0}^{\infty} \underline{w}\cdot \underline{P}_\nu L_\nu(\tfrac{1}{2}w^2) \right\}. \tag{16}$$

In this, $\underline{u}(t)$ is the drift velocity corresponding to the particle acceleration $\underline{a}(t)$. A nondimensional velocity is used:

$$\underline{w} = \frac{\underline{v} - \underline{u}(t)}{v_e} \tag{17}$$

$$f_0(\underline{w}) = (2\pi)^{-3/2} n_e e^{-\frac{1}{2}w^2} \tag{18}$$

$f(\underline{v})$ and $f_o(\underline{w})$ are normalized with respect to integration over their arguments.

The quantities $L_\nu(\frac{1}{2}w^2)$ are a set of orthogonal polynomials, chosen to be Sonine polynomials (Laguerre polynomials of order $\frac{3}{2}$). They have the property

$$\int_0^\infty dx\, e^{-x} x^{\frac{3}{2}} L_\mu(x) L_\nu(x) = \frac{(\mu+\frac{3}{2})!}{\mu!}\,\delta_{\mu\nu} \tag{19}$$

The first three are

$$L_0(t) = 1, \quad L_1(t) = \frac{5}{2} - t, \quad L_2(t) = \frac{35}{8} - \frac{7}{2}t + \frac{1}{2}t^2. \tag{20}$$

The actual drift speed can in principle be different from $\underline{u}(t)$. The difference is given by

$$\frac{1}{n}\int d^3w\, w_i\, f(\underline{w}) = \frac{1}{2}\int d^3w \sum_{\nu=0}^{\infty} P_{\nu j}\, w_i w_j\, L_\nu(\tfrac{1}{2}w^2)\, f_0(\underline{w}) \tag{21}$$

$$= \frac{1}{n}\sum_{\nu=0}^{\infty} P_{\nu i} \int d^3w\, w_i^2 L_\nu(\tfrac{1}{2}w^2)\, f_0(\underline{w})$$

$$= \sum_\nu P_{\nu i} \int dt\, (2t)^{3/2}\, \frac{4\pi}{3} L_\nu(t) (2\pi)^{-\frac{3}{2}} e^{-t}$$

$$= \sum_\nu P_{\nu i}\, \frac{4\pi}{3} \pi^{-\frac{3}{2}} \int_0^\infty dt\, e^{-t} t^{\frac{3}{2}} L_\nu(t)$$

$$= P_{0i}\,. \tag{22}$$

In this chapter, we are computing the electron thermal conductivity with the constraint that there is no net current; which is to say that we set

$$\underline{P}_0 = 0. \tag{23}$$

By doing this, we will reach a "thermoelectrically compensated" thermal conductivity without an intermediate "uncompensated" value.

If there is no net drift speed, then the thermal flux is the same whether it is computed in the rest system and then time-averaged, or

just computed in the drift system. It is

$$\mathcal{S}_i = \int d^3w \, \tfrac{1}{2} w_i w^2 f_e = \tfrac{1}{2} \sum_{\nu=1}^{\infty} P_{\nu j} \int d^3w \, w_i w_j w^2 L_m f_0(w)$$

$$= \tfrac{1}{2} \sum_{\nu} P_{\nu i} \frac{4\pi}{3} (2\pi)^{-\frac{3}{2}} 2^{5/2} \int_0^{\infty} dt \, t^{5/2} e^{-t} L_m(t)$$

$$= -\tfrac{5}{2} P_{1i} . \tag{24}$$

Thus the problem of this chapter is to compute $\underline{P}_1$.

If $f(\vec{x},\vec{v},t) = f(\vec{x},\vec{v}-\vec{u}(t))$, then the left-hand side of the Boltzmann equation (15) reduces to the gradient term. To first order in μ, only f_o contributes. By noticing that

$$\nabla w^2 = -w^2 \nabla \ln \theta \tag{25}$$

we can write

$$\nabla \ln f_0(x, v, t) = -\tfrac{3}{2} \nabla \ln \theta + \nabla \ln n_e + \tfrac{1}{2} w^2 \nabla \ln \theta$$

$$= L_0(\tfrac{1}{2}w^2) \nabla \ln n_e \sqrt{\theta} - L_1(\tfrac{1}{2}w^2) \nabla \ln \theta. \tag{26}$$

Time averages are taken by replacing $\underline{v}$ by $v_e \underline{w} + \underline{u}(t)$ and averaging over the remaining time dependence. Thus $\overline{v \cdot \nabla f} \rightarrow v_e \underline{w} \cdot \nabla f_0$ to lowest order in μ, and the LHS of (15) is

$$v_e \underline{w} \cdot \left[L_0 \nabla \ln n_e \sqrt{\theta} - L_1 \nabla \ln \theta \right] f_{e0}(w). \tag{27}$$

Klimontovich and Puchkin (1974) applied such a time average to (II.29) and give as the result (II.74), recopied here in the form

$$\overline{I}_{ei} = \frac{\partial}{\partial w_r} \int d^3k \, K_{rs} \frac{\partial f_e}{\partial w_s} \tag{28}$$

$$K_{rs}(\underline{k}, \underline{w}) = \frac{2 e^4 n Z}{m^2 v_e^3} \sum_{n=-\infty}^{\infty} \frac{k_r k_s}{k^4} J_n^2(\underline{k} \cdot \underline{d}_e) \frac{\delta(n\omega_0 - v_e \vec{k} \cdot \vec{w})}{|\epsilon(n\omega_0, k)|^2} \tag{29}$$

4. Solution for the Heat Flux

The coefficients of the L_n on the left of (27) can be isolated by operating on (15) with $\int d^3w w_i L_\mu(\frac{1}{2}w^2)$. The result is

$$\left(\nu_e \delta_{\mu 0} \nabla \ln n_e \sqrt{\theta} - \nu_e \tfrac{5}{2}\delta_{\mu 1} \nabla \ln \theta\right)_j = -\int d^3k\, d^3w\, K_{rs} \frac{\partial w_j L_\mu}{\partial w_r} \frac{\partial f_e}{\partial w_s} + \int d^3w\, w_j L_\mu\, I_{ee} \quad (30)$$

The next few steps show how to find the wanted coefficient $\underline{P}_1$ in terms of $\nabla\theta$.

In (30), we use (16) with $\nu \in \{1,2\}$ only. The integration of the delta function in (30) is simple in a coordinate system chosen so that $\underline{k} = (0,0,k)$. Vector components in this, the "internal" coordinate system, are related to those in the "external" coordinate system, which is fixed in space, by

$$x_i^E = R_{ij}\, x_j^I \quad (31)$$

where $\underline{\underline{R}}$ has the properties

$$\underline{\underline{R}} = \underline{\underline{R}}(\underline{k}),\quad R_{ij} R_{kj} = \delta_{ik} \quad (32)$$

and

$$k_i^E = R_{ij}\, k_j^I = k R_{i3}. \quad (33)$$

The external coordinate system is chosen with the third axis parallel to the driving field. To allow expressing all vector components in the integrand of (30) with respect to the internal coordinate system, we put

$$\underline{P}_\nu \cdot \underline{w} = P_{\nu k}^E w_k^E = P_{\nu k}^E R_{kl} w_l^I \quad (34)$$

The vector w_j in (30) is pinned to the external CS, so

$$w_j = R_{jm}^{E-I} w_m^I.$$

With these changes, the first integral in (30) is

$$-\int d^3k\, d^3w\, R_{k\ell} R_{jm} K_{33} \frac{\partial w_i L_\mu}{\partial w_3} \sum_{\nu=1}^{2} P_{\nu k} \frac{\partial f_0 w_\ell}{\partial w_3}. \tag{35}$$

<u>**Definition**</u>:

$$\left[H^{ik}_{\mu\nu}\right]^i = \int d^3k\, d^3w\, R_{im} R_{k\ell} K_{33} \frac{\partial w_i L_\mu}{\partial w_3} \frac{\partial f_0 w_\ell}{\partial w_3} \tag{36}$$

$$H^{ik}_{\mu\nu} = \left[H^{ik}_{\mu\nu}\right]^i + H^e_{\mu\nu}\delta_{ik} \tag{37}$$

where H^e is the value contributed by I_{ee}. This notation shortens (30) to

$$\frac{5}{2}\nu_e\left(\frac{\nabla\theta}{\theta}\right)_i \delta_{\mu 1} = \sum_\nu H^{ik}_{\mu\nu} P_{\nu k} \tag{38}$$

which is to say

$$\frac{5}{2}\nu_e\left(\frac{\nabla\theta}{\theta}\right)_i = H^{ik}_{11} P_{1k} + H^{ik}_{12} P_{2k} \tag{39}$$

$$0 = H^{ik}_{21} P_{1k} + H^{ik}_{22} P_{2k} \tag{40}$$

The tensors $\overleftrightarrow{H}_{\mu\nu}$ can be represented in the form (10):

$$H^{ik}_{\mu\nu} = \nu_0 Ⓘ_{\mu\nu}[\delta_{ik} - \delta_{i3}\delta_{k3}] + \nu_0 Ⓘ\!Ⓘ_{\mu\nu}\delta_{i3}\delta_{k3}, \tag{41}$$

where two ideograms are introduced. In terms of them, $\underline{P}_1$ is given by

$$\frac{5}{2}\frac{\nu_e}{\nu_0}\left(\frac{\nabla\theta}{\theta}\right)_3 = Ⓘ\!Ⓘ_{11} P_{13} + Ⓘ\!Ⓘ_{12} P_{23} \tag{42a}$$

$$0 = Ⓘ\!Ⓘ_{21} P_{13} + Ⓘ\!Ⓘ_{22} P_{23} \tag{42b}$$

$$\frac{5}{2}\frac{\nu_e}{\nu_0}\left(\frac{\nabla\theta}{\theta}\right)_\perp = Ⓘ_{11} P_{1\perp} + Ⓘ_{12} P_{2\perp} \tag{43a}$$

$$0 = Ⓘ_{21} P_{1\perp} + Ⓘ_{22} P_{2\perp} \tag{43b}$$

with solutions

$$P_{23} = \frac{5}{2}\frac{\nu_e}{\nu_0}\left(\frac{\nabla\theta}{\theta}\right)_3 \frac{\parallel_{22}}{\parallel_{11}\parallel_{22} - \parallel_{12}\parallel_{21}} \tag{44}$$

$$P_{1\perp} = \frac{5}{2}\frac{\nu_e}{\nu_0}\left(\frac{\nabla\theta}{\theta}\right)_\perp \frac{\perp_{22}}{\perp_{22}\perp_{22} - \perp_{12}\perp_{21}} . \tag{45}$$

The remaining task is evaluation of the components of $\overleftrightarrow{\underline{\underline{H}}}$.

5. Evaluation of $\{H^{ik}_{\mu\nu}\}$; Explicit Result for the Conductivity

The electron contribution:

Values for $H^{e}_{\mu\nu}$ in (37) are given by Landshoff (1949):

$$\{H^{e}_{\mu\nu}\}_{\mu,\nu=1,2} = \nu_0\sqrt{\pi}\begin{bmatrix} 1 & 3/4 \\ 3/4 & 45/16 \end{bmatrix} \tag{46}$$

Algorithm for evaluation of the integral:

From (36),

$$[H^{ik}_{\mu\nu}]^{i} = \int d^3k\, \mathcal{R}_{jm}\mathcal{R}_{k\ell}\int d^3w\, \mathcal{K}_{33} f_0 \frac{\partial w_j L_\mu}{\partial w_3}\Big[\frac{\partial w_\ell L_\nu}{\partial w_3} - w_3 w_\ell L_\nu\Big]. \tag{47}$$

The quantities (47) have the form

$$\mathcal{J} = \int d^3k\, \mathcal{R}_{jm}\mathcal{R}_{k\ell}\int d^3w\, \mathcal{K}_{33} f_0\, \phi_{\ell m}(\underline{w}). \tag{48}$$

$$\phi_{\ell m}(\underline{w}) = \frac{\partial w_j L_\mu}{\partial w_3}\Big[\frac{\partial w_\ell L_\nu}{\partial w_3} - w_3 w_\ell L_\nu\Big] \tag{49}$$

$$f_0 = n_e(2\pi)^{-\frac{3}{2}} e^{-\frac{1}{2}w^2} \tag{50}$$

$$\mathcal{K}_{33} = \frac{2e^4 nZ}{m^2 v_e^2} k^{-2}\sum_n J_n^2 \frac{1}{kv_e}\,\frac{\delta(w_3 - \frac{n\omega_0}{kv_e})}{|\epsilon(n\omega_0,k)|^2} \tag{51}$$

$$\mathcal{J} = \frac{2e^4 nZ}{mv_e\theta}(2\pi)^3 \int\frac{d^3k}{(2\pi)^3} k^{-3}\mathcal{R}_{i\ell}\mathcal{R}_{km}\int dw\, w^2\int d\Omega_w f_0\phi_{\ell m} w^{-1}\sum_n J_n^2\,\delta\Big(\mu_w - \frac{n\omega_0}{kv_e w}\Big) \tag{52}$$

Change of variables: $t = \frac{1}{2}w^2;\quad \eta n^2 = \frac{1}{2}\left(\frac{n\omega_0}{kv_e}\right)^2$ (53)

Abbreviation: $\nu = \eta n^2$

$$\mathcal{J} = \frac{2e^4 nZ}{mv_e\theta}(2\pi)^3(2\pi)^{-\frac{1}{2}}\int\frac{d^3k}{(2\pi)^3}k^{-3}\mathcal{R}_{i\ell}\mathcal{R}_{km}\int_\nu^\infty dt\, e^{-t}\int\frac{d\psi}{2\pi}\sum_n J_n^2\,[\phi_{\ell m}]_{w_3^2=2\nu,\ w^2=2t} \tag{54}$$

Replacing the $\mathcal{R}$ product by its azimuthal average allows use of the recipe (I.7) for the part preceding the $\mathcal{R}$ product, and it becomes

$$3\nu_0\int_0^1 d\mu_k.$$

The part of (53) after the $\mathcal{R}$'s is, with $\tilde{\phi}$ meaning $\displaystyle\int\frac{d\psi_w}{2\pi}[\phi]_{w_3^2=2\nu,\ w^2=2t}$

$$\sum_n \zeta_n^2 \int_V^\infty dt\, e^{-t} \phi_{\ell m}^{nm}(t,v) = \sum_n \zeta_n^2 e^{-v} \int_0^\infty d\tau\, e^{-\tau} \phi_{\ell m}^{nm}(\tau+v,v). \tag{55}$$

The effect of $\int d\tau\, e^{-\tau}$ on the polynomial ϕ^{nm} is to convert τ^g to $(g!)$, and the effect of $\sum_n \zeta_n^2 e^{-v}$ is to convert v^r to Γ_r. Thus

$$\mathcal{F} = 3\nu_0 \int_0^1 d\mu \left[\int \frac{d\varphi}{2\pi} R_{i\ell} R_{km}\right] Q_{\ell m}(\Gamma(\mu E)), \tag{56}$$

where $Q_{\ell m}$ is obtained from ϕ by this procedure:

(1) Take the azimuthal average.

(2) Make the replacement $w_3^2 = 2v$, $w^2 = 2t$.

(3) Make the replacement $t = \tau + v$.

(4) Make the replacement $\tau^n \to (n!)$. The result is $Q(v)$.

(5) Make the replacement $v^r \to \Gamma_r$ to get $Q(\Gamma)$.

The result of applying the algorithm will have the form

$$E_{\mu\nu}\delta_{\ell m} + F_{\mu\nu}\delta_{\ell 3}\delta_{m3} = \left\{\frac{\partial w_\ell L_\mu}{\partial w_3}\left[\frac{\partial w_m L_\nu}{\partial w_3} - w_3 w_m L_\nu\right]\right\} \text{with algorithm applied.} \tag{57}$$

Because the effect of an average over the $\underline{k}$-azimuth is

$$k_i k_k \longrightarrow \mu^2 \delta_{i3}\delta_{k3} + \tfrac{1}{2}(1-\mu^2)(\delta_{ik} - \delta_{i3}\delta_{k3}), \tag{58}$$

we have from (47) and (56) with (57)

$$\left[H_{\mu\nu}^{ik}\right]^i = 3\nu_0 \int_0^1 d\mu \left\{E_{\mu\nu}\delta_{ik} + F_{\mu\nu}\left[\mu^2\delta_{i3}\delta_{k3} + \tfrac{1}{2}(1-\mu^2)(\delta_{ik} - \delta_{i3}\delta_{k3})\right]\right\}. \tag{59}$$

That is to say

$$\text{Ⓐ}_{\mu\nu}^i = 3\int_0^1 d\mu \left\{E_{\mu\nu} + \tfrac{1}{2}(1-\mu^2) F_{\mu\nu}\right\} \tag{60}$$

$$\text{Ⓑ}_{\mu\nu}^i = 3\int_0^1 d\mu \left\{E_{\mu\nu} + \mu^2 F_{\mu\nu}\right\} \tag{61}$$

Evaluation of H in terms of Γ symbols:

The evaluation of (56) will be sketched here. The form for (57) is

$$(L_\mu \delta_{\ell 3} - B_\mu w_\ell w_3)(L_\nu \delta_{m3} - C_\nu w_m w_3)$$

$$= \left\{ L_\mu L_\nu - B_\mu L_\nu w_3^2 - C_\nu L_\mu w_3^2 + w_3^2 B_\mu C_\nu \left(\tfrac{3}{2} w_3^2 - \tfrac{1}{2} w^2\right) \right\} \delta_{\ell 3} \delta_{m3} + \tfrac{1}{2}(w^2 - w_3^2) B_\mu C_\nu w_3^2 \delta_{\ell m} \qquad (62)$$

$$B_1 = 1 \qquad B_2 = 1 + L_1 \qquad C_\nu = L_\nu + B_\nu$$

Using the expressions for B_μ and C_ν gives the intermediate result

$$\{E_{\mu\nu}\} = 2v(t-v) \begin{bmatrix} 1 + L_1(t) & 1 + L_1 + L_2 \\ (1 + L_1)^2 & (1+L_1)(1+L_1+L_2) \end{bmatrix} \qquad (63)$$

$$\{F_{\mu\nu}\} = \begin{bmatrix} L_1^2 & L_2 L_1 \\ L_1 L_2 & L_2^2 \end{bmatrix} - 2v \begin{bmatrix} L_1^2 + 2L_1 & (1+L_1)(L_1+L_2) \\ (1+L_1)(L_1+L_2) & L_2(2+2L_1+L_2) \end{bmatrix} \qquad (64)$$

$$-2v(t-3v) \begin{bmatrix} 1+L_1 & 1+L_1+L_2 \\ (1+L_1)^2 & (1+L_1)(1+L_1+L_2) \end{bmatrix} \qquad (64)$$

Substitution of (20), the explicit forms for L_ν, leads to the explicit t-dependence:

$\{E_{\mu\nu}\} = 2v(t-v)$ (65)

11	21	12	22
7/2	49/4	63/8	441/16
-1	-7	-9/2	-189/8
	1	1/2	25/4
			-1/2

The first column of (65) is to be read $E_{11} = 2v(t-v)(\frac{7}{2} - t)$.

$$\left\{F_{\mu\nu}\right\} = \begin{array}{ccc} 11 & 21 \,\&\, 12 & 22 \\ \left|\begin{array}{c} 25/4 \\ -5 \\ 1 \\ \\ \\ \end{array}\right. & \left|\begin{array}{c} 175/16 \\ -105/8 \\ 19/4 \\ -1/2 \\ \\ \end{array}\right. & \left|\begin{array}{c} 1225/64 \\ -245/8 \\ 133/8 \\ -7/2 \\ 1/4 \end{array}\right| \end{array} -2v \begin{array}{ccc} 11 & 21 \,\&\, 12 & 22 \\ \left|\begin{array}{c} 45/4 \\ -7 \\ 1 \\ \\ \\ \end{array}\right. & \left|\begin{array}{c} 385/16 \\ -181/8 \\ 25/4 \\ -1/2 \\ \\ \end{array}\right. & \left|\begin{array}{c} 3185/64 \\ -511/8 \\ 217/8 \\ -9/2 \\ 1/4 \end{array}\right| \end{array}$$

$$- 2v(t-3v) \frac{\left\{E_{\mu\nu}\right\}}{2v(t-v)} \tag{66}$$

Replacing t by $\tau+v$ and then τ^n by $n!$ gives polynomials in v:

$$\left\{E_{\mu\nu}\right\} = \begin{array}{cccc} 11 & 21 & 12 & 22 \\ \left|\begin{array}{c} 0 \\ 3 \\ -2 \\ \\ \\ \end{array}\right. & \left|\begin{array}{c} 0 \\ 17/2 \\ -12/2 \\ 4/2 \\ \\ \end{array}\right. & \left|\begin{array}{c} 0 \\ 15/4 \\ -20/4 \\ 4/4 \\ \\ \end{array}\right. & \left|\begin{array}{c} 0 \\ 93/8 \\ -122/8 \\ 52/8 \\ -8/8 \end{array}\right| \end{array} \tag{67}$$

The first column in (67) is to be read $E_{11} = 3\Gamma_1 - 2\Gamma_2$, and its precursor was the polynomial $3v - 2v^2$.

$$\left\{F_{\mu\nu}\right\} = \begin{array}{cccc} 11 & 21 & 12 & 22 \\ \left|\begin{array}{c} 13/4 \\ -74/4 \\ 92/4 \\ -24/4 \\ \\ \\ \end{array}\right. & \left|\begin{array}{c} 69/16 \\ -592/16 \\ 1032/16 \\ -512/16 \\ 80/16 \\ \\ \end{array}\right. & \left|\begin{array}{c} 69/16 \\ -516/16 \\ 832/16 \\ -400/16 \\ 48/16 \\ \\ \end{array}\right. & \left|\begin{array}{c} 433/64 \\ -3986/64 \\ 8920/64 \\ -6320/64 \\ 1744/64 \\ -160/64 \end{array}\right| \end{array} \tag{68}$$

Special values of v can be used to check the algebra: Using $v = 0$ in (65) and (66) gives the top lines of (67) and (68). The top line of (68) is the same set of numbers as Landshoff's for zero field; notice that a constant value of F will pass unchanged through (60) and (61).

Setting v = 1 in (65) and (66) leads to

$$\{E_{\mu\nu}\} = \begin{bmatrix} 1 & -1/4 \\ 9/2 & 15/8 \end{bmatrix} \qquad \{F_{\mu\nu}\} = \begin{bmatrix} 9/4 & 33/16 \\ 77/16 & 631/64 \end{bmatrix} \tag{69}$$

correctly predicting the column sums in (67) and (68). Similarly, putting v = -1 in (65) and (66) leads to

$$\{E_{\mu\nu}\} = -\begin{bmatrix} 5 & 39/4 \\ 33/2 & 275/8 \end{bmatrix} \qquad \{F_{\mu\nu}\} = \begin{bmatrix} 203/4 & 1865/16 \\ 2285/16 & 21563/64 \end{bmatrix} \tag{70}$$

agreeing with the corresponding reductions of (67) and (68).

<u>Result for the conductivity</u>:

The expressions for $K_\perp$ and $K_\parallel$ follow from comparing (9) and (24) with (44) and (45):

$$\frac{K_\perp}{K_0} = \frac{5}{2}\,\frac{0_{22}}{1_{11}\,1_{22} - 1_{12}\,1_{21}} \tag{71}$$

and the corresponding expression for $K_\parallel$. The quantities $\underline{\underline{1}}$, $\underline{\underline{0}}$ are given by (59), (60), and (61) with (46), (67), and (68).

<u>Thermal conductivity in zero field</u>:

At E = 0, the only contribution is from $\Gamma_0(0) = 1$, and (71) becomes

$$K_\perp = K_\parallel = K_0\frac{5}{2}\left[\left(\frac{13}{4}+\sqrt{2}\right) - \left(\frac{69+12\sqrt{2}}{16}\right)^2 \Big/ \left(\frac{433+180\sqrt{2}}{64}\right)\right]^{-1}$$

$$= K_0\frac{5}{2}\Big/\left[4.664 - 2.687\right] = K_0\frac{5}{2}\Big/1.977 \tag{72}$$

$$= 1.26\,K_0$$

Use of the thirteen-moment representation of Grad for the distribution function is equivalent to retaining only the first term in the denominator of (72), and is thus unsatisfactory for computing the thermal conductivity.

CHAPTER V

CONCLUSION

Ich glaube, die Wellen verschlingen
Am Ende Schiffer und Kahn;
Und das hat mit ihrem Singen
Die Lorelei getan.

Heinrich Heine[1]

Three transport coefficients have been calculated as functions of field strength for a plasma driven by an electric field with a frequency much greater than the electron-ion collision frequency, and with a strength which does not drive the electrons relativistic but is not constrained to drive them at less than the thermal speed. The technique used, which involves application of the Klimontovich formalism to evaluate the collision term and then, at an appropriate place, expanding an integrand in inverse powers of the wave number, is applicable to the calculation of other transport properties of the driven plasma.

There are several ways in which the calculation could be generalized. Inclusion of an ambient magnetic field is conceptually simple and might be analytically tractable. Other transport coefficients, the (DC) electrical conductivity in particular, could be calculated. Allowing a more general polarization for the electric field is probably simple and direct; in particular, the response to the electric field is simpler

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[1] From "Lorelei."

for a circular polarization than for linear polarization because the driven speed of an electron is constant. Including relativistic effects, by perturbation or otherwise, would extend the permitted field strength. The use of a quantum-mechanical kinetic equation is an intriguing concept; it is known to be feasible for calculating inverse Bremsstrahlung. A quantum-mechanical formulation might allow the application to more general statistics. The calculation of the second terms, of order unity, in the several expansions which begin with ln Λ ; and, in particular, including powers of k^{-2} past the first in the expansions beginning $k^{-1}(k^{-2}+\ldots)$; would give a quantitative estimate of the limits of validity of the results.

The polishing and re-organizing of the problem for simplest presentation and greatest usefulness of the results is much less complete than the author would like. All of the topics mentioned above as possible extensions might have been followed to at least the first calculational difficulty, with an expectation of including most of them. To give a general expression for integrals of the collision term and express the particular results as special cases would have been more pleasing. In the results for the thermal conductivity, there appear several linear combinations of the quantities $\Gamma_r(E)$; most of these combinations have the property that the asymptotic behavior is E^{-3} even though the individual Γ's go to $\alpha_r E^{-1}+\beta_r E^{-3}+\ldots$, because the leading terms cancel. This behavior suggests the existence of an organizing principle not recognized in the calculation. The expression for the inverse Bremsstrahlung rate can be reduced to a reasonably simple form before use of the large-k limit; it was omitted

because the result isn't new. A better overview of the Klimontovich formulation and a comparison with, e.g., the BBGKY hierarchy, could have been included. Is there a paradox lurking in the frequency-ratio factor in the Coulomb logarithm, and how is the transition made to the vanishing-field value? Does the frequency ratio explain the discordant result of White, Kilkenny, and Dangor?

CHAPTER VI

LIST OF SYMBOLS

When a page number is given, it locates the definition in the test, or shows where the symbol is used. Most of the various integer indices and dummy variables are not mentioned.

<u>Non-alphabetical</u>

$\langle\ \rangle$	Ensemble average, p. 14.
$\underline{k}$, $\vec{k}$	These two labels for a vector are used interchangeably.
$\overline{A}$	(1) Average with respect to distribution function.
	(2) Average of A over a period of the driving field.
$\left(\frac{\partial f}{\partial t}\right)_{coll}$	A label for the part of $\frac{\partial f}{\partial t}$ attributable to noncollective interparticle interaction, p. 13.
$\underline{\underline{M}}$	Hermitian conjugate of the matrix $\underline{\underline{M}}$.
$\mathrm{I}_{\mu\nu}$, $\mathrm{II}_{\mu\nu}$	Components of the matrix $(\overleftrightarrow{\underline{\underline{H}}}/\nu_0)$, p. 54.

Roman

$\underline{a}^{Mic}(\underline{x},\underline{v},t)$ "Microscopic" acceleration. The exact acceleration of a particle with coordinates ($\underline{x}$, $\underline{v}$, t) in a particular member of the ensemble. Used p. 14.

$\underline{a}$ Ensemble average of $\underline{a}^{Mic}$, p. 14.

a_i A particle acceleration corresponding to imposed force. Used p. 14.

b_o Classical distance of closest approach, p. 7.

c Speed of light.

d_α Excursion distance, $d_\alpha = \frac{g_\alpha E_0}{m_\alpha \omega_0^2}$, p. 22.

E (1) Nondimensionalized field strength, p. 11.

(2) An unspecified field-strength variable.

E_o Amplitude of the sinusoidal electric field, p. 21.

$E_{\mu\nu}(\mu E)$, $F_{\mu\nu}(\mu E)$ Functions which appear in the integrands of expressions leading to thermal conductivity, pp. 57 and 59.

e Proton charge.

$f^{Mic}(\underline{x},\underline{v},t)$ "Microscopic" distribution function. The exact distribution function for a particular set of point particles, as distinguished from the ensemble average. P. 13.

$f(\underline{xv}t)$ Distribution function.

$f(t)$, $f(tt')$ Arbitrary or unspecified function.

$G(\underline{x},\underline{x}',\underline{v},\underline{v}',t,t')$ Part of a Green's function, p. 15.

$G_{\underline{k}}(\underline{v},\underline{v}',t,t')$ Part of a Green's function, p. 16.

$I_{\alpha}(\underline{v}t)$ or $I(\underline{v}t)$ Collision term in the kinetic equation for species α or for unspecified species, pp. 13 and 17.

I (1) The energy flux of an electromagnetic wave.

(2) Collision term.

$I\{\phi(\underline{U})\}$ Momentum integral of $\phi(\underline{U})$ weighted by the collision term, p. 28.

$I_n(x)$ Bessel function, used p. 10.

$J_n(x)$ Bessel function.

$\underline{J}$ Electrical current, p. 3.

J, J_e Column vector with elements $J_n(\underline{k}.\underline{d}_e)$, p. 31.

J_i, L Column vector with elements $J_n(\underline{k}.\underline{d}_i)$, p. 31.

K (1) Thermal conductivity, p. 3.

(2) Thermoelectrically compensated thermal conductivity, p. 49 ff.

K_o Thermal conductivity, reference value, p. 46.

$K_{\perp}$, $K_{\parallel}$ Components of thermal conductivity perpendicular and parallel to the field, p. 47.

$\underline{\underline{K}}$ Susceptibility, p. 34.

$\underline{k}$ Fourier transform variable, p. 19.

k_D The "Debye wave number" $\frac{1}{\lambda_D}$. $k_{D,\alpha} = \left(\frac{4\pi n_\alpha g_\alpha^2}{\theta_\alpha}\right)^{1/2}$.

k_{max}, k_{min} Empirical limits on k, p. 5.

$\mathcal{K}_{rs}$ Weight function in the collision integral, p. 52.

L — Liouville operator, p. 16.

$L_n^{(\ell)}(x)$ — Sonine polynomials, p. 51.

$\underline{\underline{M}}$ — Dielectric matrix, p. 29 and 34.

$M(a, b, -2\zeta^2)$ — Confluent hypergeometric function, p. 10.

M — Same as m_i.

m — Mass of a particle of unspecified species, or of an electron.

m_α — Mass of a particle of species α.

$\underline{\underline{N}}$ — Inverse of dielectric matrix, p. 31.

n, n_α — Number density for unspecified species or electrons or species α.

$\mathcal{O}$ — The "order" symbol. $y = \mathcal{O}\, x$ means that $\underset{x \to 0}{\mathrm{lub}} \left| \frac{y}{x} \right|$ is finite.

$\underline{P}$ — Momentum corresponding to the deviation velocity $\underline{U}$.

P — Pressure of one species, $P = n\theta$.

$\underline{\underline{P}}$, P_{ij} — Stress tensor. $P_{ij} = \int d^3v\, m(\underline{v}-\underline{u})_i (\underline{v}-\underline{u})_j f$.

P°_{ik} — Traceless stress tensor.

$\underline{P}_\nu$ — Coefficients in an expansion in Sonine polynomials, p. 51.

$\underline{\underline{\mathcal{R}}}$ — Rotation operator, p. 53.

$\underline{S}$, $\underline{S}_\alpha$ — Thermal flux, p. 3.

$\mathcal{S}$, $\mathcal{S}_\parallel$, $\mathcal{S}_\perp$ — Dimensionless thermal flux, p. 46; components, p. 47.

$S_\alpha(\underline{xvt})$ — Source term, pp. 15 and 17.

$\underline{\underline{S}}$ — Shift operator, p. 34.

S — Same as $\underline{\underline{S}}$.

t_o — An initial time. It precedes greatly any other times defined by the stationary system, but is recent enough that collisions have not changed the distribution function.

$\underline{U}$	Velocity relative to drift, p.28 .
$\underline{u}_{\alpha o}$	Amplitude of driven velocity, p. 22.
$\underline{u}(\underline{x},t)$	Drift velocity, p. 28.
V_{eff}	P. 7.
$\underline{v}$	velocity.
v_e, v	Thermal speed, $v_\alpha = \sqrt{\frac{\theta_\alpha}{m_\alpha}}$
v_D, v_E	Electron driven speed, $v_E = \frac{eE_o}{m\omega_o}$.
$\underline{w}$	Dimensionless velocity, p. 50.
$\underline{x}^o(\underline{x},\underline{v},t)$	Orbit coordinate, p. 15.
$\underline{x}_i(t)$	Space coordinate of the i^{th} particle, p.13 .
Z	Charge number for species α . $Z_i = Z$, $Z_e = 1$.
Z	Ion charge number.

Greek

α	Fine structure constant, used p. 7.
α, β	(1) Species subscript, e or i.
	(2) A thermoelectric coefficient, p. 3.
α'	The species different from α'.
$\Gamma_r(\zeta)$	A function which appears in the analysis in Chapters III and IV, pp. 8 and 10.
$\Delta_\alpha tt'$	Velocity acquired between times t' and t, p. 23.
δ	Operator on a field quantity which defines a fluctuation, p. 14.
$\delta \underline{a}(\underline{x}, \underline{v} t)$	Defined on p. 14 if each letter f is replaced by $\underline{a}$.
$\delta \underline{E}$	An electric field related to $\delta \underline{a}$ by $\delta \underline{a}(xvt) = \frac{e_\alpha}{m_\alpha} \delta \underline{E}(xvt)$.
δf°	P. 17.
$\delta \vec{J}$	$\int d\mathbf{v}\, e_\alpha \vec{v}\, \delta f$, p. 30.
$\delta \rho^\circ$	P. 18.
ϵ	P. 18.
ϵ_n	Watson's notation; $\epsilon_n = 2 - \delta_{n,0}$, used only in Appendix A.
ξ, η	Convenience variables, p. 11.
$\Theta(t)$	The Heaviside step function, zero for negative arguments and unity for positive arguments.
θ, θ_α	Temperature in energy units.
Λ	Argument of Coulomb logarithm, p. 5.
Λ_n	Bessel function, p. 10.
λ_o	The Debye shielding distance. See k_D.
λ_{deB}	de Broglie wavelength of a thermal electron, p. 8.

μ (1) Dimensionless parameter proportional to the gradient, p. 45.

(2) A spherical coordinate, cosine of the colatitude.

(3) In Chapter IV, an integer subscript.

ν_o Electron-ion collision frequency, p. 7. Often in the literature called ν_{ei}.

ν_θ Effective collision frequency for heat exchange, p. 43.

σ (1) A thermoelectric coefficient, p. 3.

(2) AC electrical conductivity, p. 29.

σ_D Dissipative (real) part of σ, p. 30.

τ_{ei} Electron-ion equilibration time, p. 3.

$\Upsilon(E)$ A parameter in the heat-exchange rate, p. 43.

ψ_α Perturbation of distribution function, p. 46.

$\psi_{\alpha\underline{k}}(\underline{v}tt')$ A phase, p. 17.

$\phi(\underline{U})$ Arbitrary function of $\underline{U}$, p. 27.

$\chi_{\alpha\underline{k}}(\omega)$ Linear susceptibility of species α, pp.24 and 11.

$\chi_{\alpha\underline{k}}(tt')$ Susceptibility contributed by species α, p. 18. There are two susceptibilities, distinguished by the number of time or frequency arguments.

χ_k $= \chi_{e\underline{k}} + \chi_{i\underline{k}}$

$\underline{\underline{\Omega}}$ Frequency matrix, p. 31.

ω Angular frequency, a Fourier transform variable.

ω_o Angular frequency of the driving field.

$\omega_p, \omega_{p\alpha}$ Plasma frequency for electrons or species α, $\omega_{p\alpha}^2 = \dfrac{4\pi n_\alpha q_\alpha^2}{m_\alpha}$.

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APPENDICES

Appendix A: Alternative Derivation for $\Gamma_r(\zeta)$

This appendix shows another derivation of (I.26) for Γ_r.

Lemma:

$$\sum_{n=-\infty}^{\infty} n^{2r} J_n^2(x) = \frac{(r-\frac{1}{2})!}{r!\,(-\frac{1}{2})!} x^{2r} + \mathcal{O}\, x^{2r-2} \qquad \text{(A.1)}$$

Proof:

From Watson (1944, p. 36), we have Neumann's expansion:

$$\left(\tfrac{1}{2}x\right)^{2r} = \frac{r!\,r!}{(2r)!} \sum_{n=r}^{\infty} n\,\epsilon_n \frac{(n+r-1)!}{(n-r)!} J_n^2(x) \qquad \text{(A.2)}$$

$$= \frac{r!\,r!}{(2r)!} \sum_{n=-\infty}^{\infty} J_n^2(x) \prod_{s=0}^{r-1} (n^2 - s^2)$$

$$= \frac{r!\,r!}{(2r)!} \sum_{n} \left[n^{2r} - \frac{1}{6} r(r-1)(2r-1) n^{2r-2} + \dots\right] J_n^2(x) \qquad \text{(A.3)}$$

Thus each sum $\sum_n n^{2r} J_n^2$ is a polynomial of order r in x^2. The highest-order term is $x^{2r} \frac{(2r)!}{2^r r!\,r!}$, which leads to (A.1).

The definition (I.10) gives

$$\Gamma_r(\zeta) = \lim_{\eta \to 0} \sum_{n=-\infty}^{\infty} (\eta n^2)^r \sum_{p=0}^{\infty} \frac{(-)^p}{p!} (\eta n^2)^p J_n^2\left(\zeta \sqrt{\frac{2}{\eta}}\right). \qquad \text{(A.4)}$$

Using the lemma makes this

$$\Gamma_r(\zeta) = \sum_{p=0}^{\infty} \frac{(-)^p}{p!} (2\zeta^2)^{p+r} \frac{(p+r-\frac{1}{2})!}{(p+r)!\,(-\frac{1}{2})!}$$

$$= \frac{(r-\frac{1}{2})!}{r!\,(-\frac{1}{2})!} (2\zeta^2)^r M(r+\tfrac{1}{2}, r+1, -2\zeta^2) \qquad \text{(A.5)}$$

which is the same as (I.26).

Appendix B: Γ_r in terms of Bessel Functions

There is a formula in Abramowitz and Stegun (1964, p. 506) which was incorrect in early printings:

$$M(a,b,z)$$

$$= e^{\frac{1}{2}z}\Gamma(b-a-\tfrac{1}{2})\left(\frac{z}{4}\right)^{a-b+\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(-)^n}{n!}I_{n+b-a-\frac{1}{2}}(\tfrac{1}{2}z)\,(b-a-\tfrac{1}{2}+n)\,\frac{(2b-2a-1)_n\,(b-2a)_n}{(b)_n} \tag{B.1}$$

In this,

$$(b)_n \equiv \frac{(n+b-1)!}{(b-1)!} \,. \tag{B.2}$$

We put $a = q + \frac{1}{2}$, $b = q + d + 1$, expecting to take the limit as d goes to zero:

$$M(q+\tfrac{1}{2}, q+d+1, z) = e^{\frac{1}{2}z}\underline{\Gamma(d)}\left(\tfrac{1}{4}z\right)^{-d}\sum_{n=0}^{\infty}\frac{(-)^n}{n!}I_{n+d}(\tfrac{1}{2}z)\underline{(n+d)\,\frac{(2d)_n(d-q)_n}{(q+d+1)_n}} \tag{B.3}$$

The factors related to the limit are underlined. If $n = 0$, they become

$$\Gamma(d)d = d! \longrightarrow 1,$$

and if $n \neq 0$,

$$\Gamma(d)\, n\, \frac{(2d+n-1)!\;(d-q+n-1)!\;(q+d)!}{(2d-1)!\;(d-q-1)!\;(q+d+n)!}$$

$$= n\frac{d!}{d}\,\frac{2d}{(2d)!}(n-1)!\;\frac{(d-q)(d-q+1)\ldots(d-q+n-1)\;q!}{(q+n)!}$$

$$= 2n!\;\frac{q!}{(q+n)!}\;(-)^n\;q(q-1)\ldots(q+1-n)$$

$$= 2n!\,(-)^n\;\frac{q!\,q!}{(q+n)!\,(q-n)!}$$

$$M(q+\tfrac{1}{2}, q+d+1, -2\zeta^2) = e^{-\zeta^2}\left\{I_0(-\zeta^2) + \sum_{n=1}^{\infty}\frac{(-)^n}{n!}I_n(-\zeta^2)\,2n!(-)^n\frac{q!\,q!}{(q-n)!\,(q+n)!}\right\}$$

$$= \sum_{n=-\infty}^{\infty}(-)^n\Lambda_n(\zeta^2)\,\frac{q!\,q!}{(q-n)!\,(q+n)!} \tag{B.4}$$

A simple check of that is available: Putting $q = -\frac{1}{2}$ makes it

$$M(0,\tfrac{1}{2},-2\zeta^2) = \sum_{n=-\infty}^{\infty} (-)^n \Lambda_n(\zeta^2) \frac{\pi}{(n-\frac{1}{2})!\,(-n-\frac{1}{2})!}$$

$$= \sum (-)^n \Lambda_n(\zeta^2) \frac{\sin \pi(n+\frac{1}{2})}{1} = \sum_n \Lambda_n(\zeta^2) \tag{B.5}$$

or

$1 = 1.$

Also in agreement is $M(\frac{1}{2},1,-2\zeta^2) = \Lambda_0(\zeta^2)$.

Combining (B.4) with the other factors of (I.26) gives,

$$\Gamma_r(\zeta) = \frac{(2r-1)!!}{r!}\, \zeta^{2r} \sum_{n=-\infty}^{\infty} (-)^n \Lambda_n(\zeta^2) \frac{r!\,r!}{(r+n)!\,(r-n)!}$$

Appendix C: Recursion Formulas

This appendix shows how recursion formulas from Abramowitz and Stegun (1964) translate to formulas for the Γ symbols. The arguments are related by $z = -2\zeta^2$.

Recursion:

$$b(-b+z)M(a,b,z) + b(b-1)M(a-1,b-1,z) - azM(a+1,b+1,z) = 0$$

Setting

$a = r+\frac{1}{2}, \quad b = r+1$

and multiplying by $\frac{(r-\frac{1}{2})!}{(-\frac{1}{2})!\,(r+1)!}(-z)^r$ gives

$$(z-r)\Gamma_r(\zeta) - z(r-\tfrac{1}{2})\Gamma_{r-1}(\zeta) + \Gamma_{r+1}(\zeta) = 0$$

$$2\zeta^2\Gamma_r(\zeta) + (2r-1)\zeta^2\Gamma_{r-1}(\zeta) = \Gamma_{r+1}(\zeta) - r\Gamma_r(\zeta)$$

Differential recursion:

$$M'(a,b,z) = \frac{a}{b} M(a+1,b+1,z)$$

$$[z^r M(a,b,z)]' = \frac{a}{b} z^r M(a+1,b+1,z) + r z^{r-1} M(a,b,z)$$

Including a factor $\dfrac{(r-\frac{1}{2})!}{r!\,(-\frac{1}{2})!} (-)^r z$ gives

$$z \frac{d}{dz} \Gamma_r = \frac{1}{2} \zeta \Gamma_r'(\zeta) = -\Gamma_{r+1}(\zeta) + r \Gamma_r(\zeta)$$

because of $z = -2\zeta^2$.

Appendix D: Initial-Time Correlation Function

If the two-particle correlation function is ignored, Eq. (42) from Wu (1966) is

$$\langle \delta f_\alpha(\underline{x}\,\underline{v}\,t)\, \delta f_\beta(\underline{x}'\underline{v}'t) \rangle = \delta_{\alpha\beta}\, n_\alpha\, \delta(\underline{x}-\underline{x}')\, \delta(\underline{v}-\underline{v}')\, F_\alpha(\underline{x}\,\underline{v}\,t)$$

Taking the Fourier transform with respect to each of $\underline{x}$ and $\underline{x}'$ gives

$$\langle \delta f_{\alpha\underline{k}}(\underline{v}t)\, \delta f_{\beta,-\underline{k}}(\underline{v}'t) \rangle = \delta_{\alpha\beta}\, n_\alpha\, \delta(\underline{v}-\underline{v}') \int d^3x\, d^3x'\, e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}\, \delta(\underline{x}-\underline{x}')\, F_\alpha(\underline{x}\,\underline{v}\,t)$$

$$= \delta_{\alpha\beta}\, n_\alpha\, \delta(\underline{v}-\underline{v}')\, F_\alpha(\underline{v}\,t)$$

in which F $(\underline{x},\underline{v},t)$ is assumed independent of $\underline{x}$, and the unit box normalization has been used. This is (II.44).

Appendix E: A Property of the Green's Function

Claim:

$$G_{\alpha\underline{k}}(\underline{v}\underline{v}'tt') = \int d^3v''\, G_{\alpha\underline{k}}(\underline{v}\underline{v}''tt_0)\, G_{\alpha,-\underline{k}}(\underline{v}'\underline{v}''t't_0)$$

Proof:

The integral is

$$\int d^3v''\, \delta(\underline{v}-\underline{v}''-\Delta_\alpha tt_0)\, e^{-i\varphi_{\alpha\underline{k}}(\underline{v}-\Delta tt_0,t,t_0)}\, \delta(\underline{v}'-\underline{v}''-\Delta_\alpha tt_0)\, e^{-i\varphi_{\alpha,-\underline{k}}(\underline{v}'-\Delta t't_0,t',t_0)}$$

Using the delta functions gives as the sum of the phases

$$\varphi_{\alpha\underline{k}}(\underline{v}-\Delta_\alpha tt_0,t,t_0) + \varphi_{\alpha,-\underline{k}}(\underline{v}'-\Delta t't_0,t',t_0)$$

$$= \underline{k}\cdot\left\{\underline{v}''(t-t_0) - \underline{v}''(t'-t_0) + \int_{t_0}^{t} d\tau\, \Delta_\alpha \tau t_0 - \int_{t_0}^{t'} d\tau\, \Delta_\alpha \tau t_0\right\}$$

$$= \underline{k}\cdot\left\{\underline{v}''(t-t') + \int_{t'}^{t} d\tau\, \Delta_\alpha \tau t_0\right\}$$

$$= \underline{k}\cdot\left\{\underline{v}'(t-t') - (t-t')\Delta_\alpha tt_0 + \int_{t'}^{t} d\tau\, \Delta_\alpha \tau t_0\right\}$$

$$= \underline{k}\cdot\left\{\underline{v}'(t-t') + \int_{t'}^{t} d\tau\, \Delta_\alpha \tau t'\right\}$$

$$= \varphi_{\alpha\underline{k}}(\underline{v}', t, t')$$

Eliminating $\underline{v}''$ from the delta functions gives the correct argument for the remaining delta function, completing the proof.

Appendix F: Rationale for the Choice of k_{max}

A crude argument is given in two steps:

Step One: The kinetic equation for the fluctuations is simplified by ignoring terms quadratic in the fluctuations. These terms are important when particles are close. It is known (vide, e.g., Krall and Trivelpiece 1973) that close collisions contribute only a fraction $\mathcal{O}(\frac{1}{\ln \Lambda})$ to the Rutherford scattering cross-section. Ignoring collisions which produce deflections greater than $\frac{\pi}{2}$ preserves the validity of the kinetic equation without losing important effects.

Step Two: Those parts of a configuration of particles which involve particles closer than a distance b are described, when the space variable is replaced by a Fourier-transform variable $\underline{k}$, by values of $\underline{k}$ satisfying

$$k \gtrsim \frac{2\pi}{b},$$

so the region correctly described by the linearized kinetic equation is the one

$$k \ll \frac{2\pi}{b_0}.$$